Section Notes 7
IP: Cutting Planes

Applied Math / Engineering Sciences 121

Week of November 4, 2019

Goals for the week

- understand what a strong formulations is.
- be familiar with the cutting planes algorithm and the types of cuts that can be generated.
- be able to generate valid inequalities and cuts using algebraic reasoning, be able to generate cover inequalities.
- be comfortable with the Chvátal-Gomory procedure and be able to find C-G cuts.

Contents

1 IP: Guidelines for Strong Formulations, Continued

1.1 An Optimal IP Formulation: The Convex Hull

We have argued that formulations for an IP are better if the corresponding polyhedron of feasible LPR solutions is smaller. To answer the question regarding an **optimal IP formulation** (i.e. the smallest possible polyhedron), we need the following definition:

Convex hull. Let $T = \{x^1, ..., x^k\}$ be the set of feasible integer solutions to a particular integer programming problem. We assume that the feasible set is bounded and, therefore, T is finite. We consider the **convex hull of** T: $CH(T) = \Big\{ \sum_{i=1}^{k} \lambda_i x^i | \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0, x^i \in T \Big\}$

We know that the set $CH(T)$ is a polyhedron with integer extreme points. If we knew $CH(T)$ explicitly, i.e., if we could represent $CH(T)$ in the form $CH(T) = \{x | Ax \leq b\}$, we could solve the integer programming problem

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & x \in T \end{aligned}$$

by finding an extreme point solution to the linear programming problem

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & x \in CH(T) \end{aligned}$$

which we know how to solve efficiently. However, it is often difficult to find a formulation whose linear programming relaxation is the convex hull $CH(T)$ of the integer feasible solutions, and such a representation may have an exponential number of constraints. In these cases, it is reasonable to strive for a compromise and to try to find a polyhedron that closely approximates $CH(T)$.

1.2 Weak Duals

Exercise 1

Remembering Duality.

1. What does the weak duality theorem for linear programming say?
2. What does the strong duality theorem for linear programming say?
3. What do we know about the relation between the primal and the dual for integer programming?

End Exercise 1

By reading the above section and completing the above exercises, you should now:

- understand the concept of optimal IP formulations
- be comfortable with relationship between the dual and primal for integer programming.

2 Cutting Planes Algorithm

2.1 Formulation Strength

Remember from last time: given two formulations P_1 and P_2 for the same integer programming (IP) problem, where P_i denotes the formulation for a problem and the corresponding polytope. We say that P_1 is a **stronger formulation** than P_2 if $P_1 \subset P_2$.

Exercise 2

1. Why do we prefer stronger formulations for IPs over weaker ones?
2. Why is the convex hull of the feasible integer solutions the best formulation for an IP we can hope for?
3. Why do we think we can solve an IP efficiently when we are given its convex hull representation?
4. Why will it generally not be helpful to search for the convex hull representation of a problem?

End Exercise 2

By completing the above exercises, you should now:

- understand what a strong formulation is.
- understand the significance of the convex hull representation of a problem.

2.2 Review: The general idea of Cutting Planes Algorithms

Main idea: If we cannot find succinct convex hull formulations in general, let's approximate the convex hull efficiently for every given instance (note: general case vs. specific instance). This leads to the idea of the cutting planes algorithm. But we need a few more definitions before we get there:

1. **Valid Inequalities**: an inequality $\pi^T x \leq \pi_0$ is a valid inequality for $X \subseteq R^n$ if $\pi^T x \leq \pi_0 \forall x \in X$.

2. **Separation Problem**: The separation problem is the problem of a) given some $X \subseteq R^n$ and $x^* \in R^n$, is $x^* \in conv(X)$? b) if not, find a valid inequality that is violated by x^*.

3. **Cut**: A cut is a valid inequality that separates the current fractional solution x^*.

4. The **cutting plane algorithm** in its general form can then be formulated as follows:
   Step 1: Solve the LPR. Get x^*.
   Step 2: If x^* integral stop, else find a valid inequality that will exclude x^*.
   Step 3: Go to Step 1.

The main challenges are then a) to find valid cuts (inequalities), b) to find cuts that will quickly lead to the optimal integer solution, c) to find a method for generating cuts that is guaranteed to terminate. These are the problems that much of the research on cutting planes algorithm addresses. We will look at three different methods for generating cuts in this section.

3 A First Look at Generating Cuts

3.1 Review: Generating Cuts

One way to generate some cuts is to come up with a valid inequality simply via algebraic reasoning. We have seen the following example in lecture before (here B denotes the set of binary integers, i.e. $B = \{0, 1\}$). Consider the following 0-1 knapsack set:

$$X = \{x \in B^5 : 3x_1 - 4x_2 + 2x_3 - 3x_4 + x_5 \leq -2\}$$

Now, let's reason about what always has to hold about the variables. If $x_2 = x_4 = 0$, then the LHS $= 3x_1 + 2x_3 + x_5 \geq 0$ and the RHS $= -2$, which is impossible. So all feasible solutions satisfy $x_2 + x_4 \geq 1$ and this is a valid inequality.

3.2 Practice: Generating Cuts

Exercise 3

1. Find a second valid inequality for the 0-1 knapsack problem given above.
2. $X = \{(x, y) \in Z \times B : 2x - 3y \leq 4\}$. Assume we found the point $x^* = (3.5, 1)$ as a solution to the LPR. Find a valid inequality for X cutting off x^*.
3. $X = \{(x, y) \in Z^2 : x \leq 2y, 2x \geq 3\}$. Assume we found the point $x^* = (1.5, 0.8)$ as a solution to the LPR. Find a valid inequality for X cutting off x^*.
4. $X = \{(x, y) \in Z^2 : x \geq 2, 4y \leq x\}$. Assume we found the point $x^* = (2, 0.5)$ as a solution to the LPR. Find a valid inequality for X cutting off x^*.

Note, the points to be cut off in these exercises were just chosen to illustrate one way to generate cuts. Those points were not necessarily optimal solutions to the LPRs.

End Exercise 3

By completing the above exercises, you should now:

- know how to use algebraic manipulations to come up with a valid inequality.

4 Cover Inequalities

4.1 Basics: Cover Inequalities

(From L. Wolsey, Ch. 9.3.1) Consider the set $X = \{x \in B^n : \sum_{j=1}^{n} a_j x_j \leq b\}$. Complementing variables if necessary by setting $\overline{x}_j = 1 - x_j$, we assume throughout this section that the coefficients ${a_j}_{j=1}^{n}$ are positive. Also we assume $b > 0$. Let $N = \{1, \ldots, n\}$.

Definition: A set $C \subseteq N$ is a *cover* if $\sum_{j \in C} a_j > b$. A cover is *minimal* if $C \setminus j$ is not a cover for any $j \in C$.

Definition: A cover provides a valid inequality called the cover inequality, which is represented as $\sum_{j \in C} x_j \leq |C| - 1$.

A set $C \subseteq N$ is a *cover* if $\sum_{j \in C} a_j > b$. A cover is *minimal* if $C \setminus j$ is not a cover for any $j \in C$.

The idea behind cover inequalities is that the coefficients of a specific set of binary decision variables add up to more than the RHS of the inequality constraint. From this we can infer that not all these decision variables can be set to 1. Let's look at an easy example first: Consider the following knapsack problem:

$$X = \{x \in B^4 : 3x_1 + 4x_2 + 5x_3 + 7x_4 \leq 11\}$$

Two minimal cover inequalities for X are:

$$x_1 + x_2 + x_3 \leq 2$$

$$x_3 + x_4 \leq 1$$

4.2 Practice: Cover Inequalities

Exercise 4

Consider the knapsack problem:

$$X = \{x \in B^7 : 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}$$

Find as many cover inequalities for this problem as you can. Are your cover inequalities minimal?

End Exercise 4

4.3 Strengthening Cover Inequalities

Although the cover inequalities certainly strengthen the formulation by constraining the problem, even stronger inequalities are desirable. We describe two simple ways to strengthen the basic cover inequality.

4.3.1 Extended Cover Inequalities

Proposition: If C is a cover for X, the extended cover inequality

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

is valid for X, where $E(C) = C \cup \{j : a_j \geq a_i \quad \forall i \in C\}$.

Let's look again at the example from before:

$$X = \{x \in B^4 : 3x_1 + 4x_2 + 5x_3 + 7x_4 \leq 11\}$$

One minimal cover inequality that we had found was:

$$x_1 + x_2 + x_3 \leq 2$$

Note that the coefficient for x_4 is larger than the coefficients of all decision variables of the initial cover. Now, notice that when $x_4 = 1$ at most one of the other variables can also have value 1. Thus, we can add x_4 to the cover to strengthen it. This is the extended cover inequality:

$$x_1 + x_2 + x_3 + x_4 \leq 2$$

4.3.2 Lifted Cover Inequalities

Given the cover inequality:

$$x_1 + x_2 + x_3 \leq 2$$

we can also try to strengthen this inequality by *lifting* to:

$$x_1 + x_2 + x_3 + \alpha_4 x_4 \leq 2 \qquad (*)$$

for some α_4. This is valid for $x_4 = 0$ because in that case it is the original cover inequality. When $x_4 = 1$ the original inequality in the 0-1 knapsack requires:

$$3x_1 + 4x_2 + 5x_3 \leq 4 \qquad (**)$$

The idea is to pick maximal α_4 such that $(*)$ holds with $x_4 = 1$ for all binary values for x_1, x_2 and x_3 that satisfy $(**)$. Setting $\alpha_4 = 2 - \max\{x_1+x_2+x_3 : 3x_1+4x_2+5x_3 \leq 4, x \in \{0,1\}^3\}$, we get $\alpha_4 = 1$.

4.4 Practice: Strengthening Cover Inequalities

Exercise 5

1. Consider the seven variable 0-1-knapsack problem:

   $$X = \{x \in B^7 : 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19\}$$

   One cover that we found was $C = \{3, 4, 5, 6\}$. Find a way to extend this cover inequality.

2. Can you find an even stronger inequality than the one you found in part (1), not by adding another variable to the inequality but by increasing the coefficient of one of the variables from 1 to 2?

3. Clearly when $x_1 = x_2 = x_7 = 0$, the inequality $x_3 + x_4 + x_5 + x_6 \leq 3$ is valid for $\{x \in B^4 : 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19\}$. Now keeping $x_2 = x_7 = 0$, for what values of α_1 is the inequality $\alpha_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$ valid for $\{x \in B^5 : 11x_1 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19\}$?

End Exercise 5

By completing the above exercises, you should now:

- understand that when coefficients of a set of binary decision variables add up to more than the RHS of the inequality constraint, not all these variables can be set to 1.
- know what cover inequalities are and how to strengthen them.

5 Chvátal-Gomory Valid Inequalities

We have seen several ways to generate valid inequalities. However, we'd like to have a general procedure that can be used to define a *complete* cutting planes algorithm (able to solve IPs by repeatedly cutting off the optimal fractional solution). Gomory's cutting plane algorithm, which uses Chvátal-Gomory valid inequalities, achieves this.

5.1 Example 1: C-G Valid Inequalities

Assume your are given the following IP:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & 2x_1 + 0.5x_2 \leq 3.5 \\ & x_1, x_2 \geq 0 \quad \text{and} \quad \textit{integer} \end{aligned}$$

We can round down the coefficient of x_2 to the nearest integer (we are only weakening the inequality which is always correct) to obtain the following valid inequality.

$$2x_1 + 0x_2 \leq 3.5$$

Now, as the LHS is integral for all points of X, we can reduce the RHS to the nearest integer, and we obtain the valid inequality for X:

$$2x_1 \leq 3$$

5.2 Review: C-G Valid Inequalities

Chvátal-Gomory procedure to construct a valid inequality for the set $X = P \cap Z^n$, where $P = \{x \in R^n_+ : Ax \leq b\}$, A is an $m \times n$ matrix with columns $\{a_1, a_2, \ldots, a_n\}$, and $u \in R^m_+$:

1. The inequality
$$\sum_{j=1}^{n} (\sum_i u_i a_{ij}) x_j \leq \sum_i u_i b_i$$
is valid for P as $u_i \geq 0$ and $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ (for all rows i)
(we can always multiply the inequalities with positive scalars and add them)

2. The inequality
$$\sum_{j=1}^{n} \lfloor \sum_i u_i a_{ij} \rfloor x_j \leq \sum_i u_i b_i$$
is valid for P as $x \geq 0$,
(we only weakened the inequality, which is always valid)

3. The inequality
$$\sum_{j=1}^{n} \lfloor \sum_i u_i a_{ij} \rfloor x_j \leq \lfloor \sum_i u_i b_i \rfloor$$
is valid for X as $x \in X$ is integer, and thus the LHS is always integer (because the coefficients are integer). This is where we use the integrality of the decision variables.

5.3 Practice 1: C-G Valid Inequalities

Suppose you are given the following IP:

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & 2x_1 + 3x_2 \leq 4 \\ & x_1 - 2x_2 \leq 1 \\ & x_1, x_2 \geq 0 \quad \text{and} \quad \textit{integer} \end{aligned}$$

and assume you have $x^* = (1.1, 0.4)$ which is a feasible LPR solution (not optimal here, but good enough to explain the idea of using a C-G valid inequality as a cut.)

Exercise 6

Let's try to find a C-G valid inequality that cuts off x^*.

End Exercise 6

5.4 Practice 2: Gomory's cutting plane algorithm

We are going to use Gomory's cutting plane algorithm. This uses C-G valid inequalities in a particular way to solve IPs.

In addition to having integer, non-negative decision variables, it's important to have integer coefficients and integer RHS values. Suppose we have the following IP:

$$\begin{aligned} \max \quad & -5x_1 - 9x_2 - 23x_3 \\ \text{s.t.} \quad & 20x_1 + 35x_2 + 95x_3 \geq 319 \\ & x_1, x_2, x_3 \geq 0 \quad \text{and} \quad \textit{integer} \end{aligned}$$

Bringing this problem to the standard equality form we have:

$$\begin{aligned} \max \quad & -5x_1 - 9x_2 - 23x_3 \\ \text{s.t.} \quad & -20x_1 + (-35x_2) + (-95x_3) + x_4 = -319 \\ & x_1, x_2, x_3, x_4 \geq 0 \quad \text{and} \quad \textit{integer} \end{aligned}$$

After a few dual-simplex iterations, the solution of the relaxed problem can be found to be $(15.95, 0, 0, 0)$ with objective 79.75. The constraint in the final tableau reads:

$$x_1 + 1.75x_2 + 1.95x_3 - 0.05x_4 = 15.95$$

where the objective line has become:

$$0.25x_2 + 13.25x_3 + 0.25x_4$$

Exercise 7

Solve the integer problem using Gomory's cutting plane algorithm.

End Exercise 7

5.5 Practice: Gomory's cutting plane algorithm

(From L. Wolsey, pages 124-125).

Assume you have solved the LPR of an IP and found an optimal solution. Let B' denote the non-basic variables.

If the basic optimal solution x^* is not integer, there exists some row i with a fractional right hand side.

Choosing such a row (i), and considering the equality as an inequality (which is valid), the Chvátal-Gomory valid inequality for this row is:

$$x_i + \sum_{j \in B'} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor$$

We need to add this to the tableau. It will be a cut, hence we call these Gomory cuts!

For this, we can rewrite this inequality by eliminating x_i. To do this, we add the original equality to the negated inequality. Eliminating x_i makes it easier to bring into the tableau in a way that leaves x_i isolated.

This gives:

$$\sum_{j \in B'} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor.$$

(This is the general short-cut for adding a CG valid inequality to a tableau, using an equality with a fractional RHS value.)

The tableau below gives the LP solution to the relaxation of an integer programming problem with maximization objective. All a_{ij} and b_i coefficients in the original problem are integer.

$$\begin{bmatrix} vars & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & RHS \\ z & 0.1 & 0.3 & 0 & 0.2 & 0 & 0 & 23.1 \\ x_6 & 1.3 & -0.3 & 0 & -1.0 & 0 & 1 & 5.3 \\ x_3 & 0 & 1.1 & 1 & 0.4 & 0 & 0 & 1.6 \\ x_5 & -0.8 & -0.2 & 0 & -0.5 & 1 & 0 & 3.7 \end{bmatrix}$$

Exercise 8

Write out all Gomory cuts that can be derived from the tableau.

End Exercise 8

Exercise 9

Add the Gomory cut with the largest right-hand-side value to the tableau and use the dual simplex method to find the new solution.

End Exercise 9

By completing the above exercises, you should now:

- understand the Chvátal-Gomory procedure for generating valid inequalities.
- be able to find Gomory cuts in the context of the Gomory cutting plane algorithm.

6 Solutions

Solution 1

1. The value of a feasible primal solution is less than or equal to the value of a feasible dual solution.
2. If an LP has an optimal solution then so does its dual and the two optimal values are equal.
3. We know that any solution for **the dual of the LPR of the primal IP** is greater than or equal to any solution to the **primal IP**, i.e. the primal IP and the dual of the LPR of the primal IP form a **weak dual pair**.)

End Solution 1

Solution 2

1. The main reason for preferring stronger formulations is that we can obtain tighter bounds when we use LP relaxations in an IP algorithm such as branch and bound. Another less important reason is that we have a smaller space of non-integer solutions and thus we are more likely to find integer solutions quickly when running branch and bound.
2. The convex hull is the smallest polytope that still includes all integer feasible solutions. In that sense, the convex hull formulation is the strongest one possible.
3. When given the convex hull formulation we can simply solve the LPR of the IP. As all corner points of the convex hull must be integer, we are guaranteed to get an integer feasible solution that is optimal. We know that we have efficient algorithms for solving linear programs (polynomial time algorithms). Note that this argument only makes sense if the convex hull formulation is succinct (i.e. polynomially sized).
4. In general, it is difficult to find the convex hull formulation of an IP. But even more importantly, in general the convex hull formulation will require an exponential number of inequalities to be described (this is, because solving IPs is NP-hard). Thus, in that case, solving the LPR of the IP would take exponential instead of polynomial time which is infeasible for most problems in practice.

End Solution 2

Solution 3

1. If $x_1 = 1$ and $x_2 = 0$, the LHS $= 3 + 2x_3 - 3x_4 + x_5 \geq 3 - 3 = 0$ and the RHS $= -2$, which is impossible, so $x_1 \leq x_2$ is also a valid inequality.

2. $2x - 3y \leq 4$ is equivalent to $2x \leq 3y + 4$. Because y is binary, we know that $2x \leq 7$ or $x \leq 3.5$. Because x is integer, we know that $x \leq 3$ which is a valid inequality that cuts off $(3.5, 1)$.

3. $x \leq 2y$ is equivalent to $2x \leq 4y$ which is equivalent to $-2x \geq -4y$. We can then add this inequality to $2x \geq 3$ and we get $0 \geq 3 - 4y$. We can transform this to $y \geq \frac{3}{4}$. Because we know that y is integer we get the valid inequality $y \geq 1$ which cuts off $(1.5, 0.8)$.

4. Consider the smallest value for $x = 2$. Then y has to be 0. Now consider $x = 3$. Then y still has to be 0. Consider $x = 4$. Now $y \leq 1$. As you can see, the difference between x and y is always at least 2. Thus, we have a new valid inequality: $x - y \geq 2$. This inequality obviously cuts of the point $x^* = (2, 0.5)$.

End Solution 3

Solution 4

Some cover inequalities for X are:

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 2 \\ x_1 + x_2 + x_6 &\leq 2 \\ x_1 + x_5 + x_6 &\leq 2 \\ x_3 + x_4 + x_5 + x_6 &\leq 3 \end{aligned}$$

All of those cover inequalities are minimal.

End Solution 4

Solution 5

1. The extended cover inequality for $C = \{3, 4, 5, 6\}$ is $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$.

2. We can strengthen the inequality even further by increases x_1's coefficient to 2: $2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$. To see this, find the max binary values for $x_2, \ldots, x_7$ that satisfy the 0-1 knapsack constraint when $x_1 = 1$ and that maximize $3 - (x_2 + x_3 + x_4 + x_4 + x_5 + x_6)$. In particular, $\alpha_1 = 3 - \max\{x_2 + x_3 + x_4 + x_4 + x_5 + x_6 : 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 8, x \in \{0, 1\}^6\}$. This is $\alpha_1 = 3 - 1 = 2$, using solution $x_6 = 1$ and all the other x values (except x_1) $= 0$.

3. When $x_1 = 0$, the inequality is known to be valid for all α_1. When $x_1 = 1$, it is a valid inequality if $\alpha_1 + x_3 + x_4 + x_5 + x_6 \leq 3$ is valid for all $x \in B^4$ satisfying

$6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19 - 11$. Equivalently if and only if $\alpha_1 \leq 3 - \eta$, where $\eta = \max\{x_3 + x_4 + x_5 + x_6 : 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 8, x \in B^4\}$. Now $\eta = 1$ at $x = (0, 0, 0, 1)$ and hence $\alpha_1 \leq 2$. Thus the inequality is valid for all values of $\alpha_1 \leq 2$, and $\alpha_1 = 2$ gives the strongest inequality.

End Solution 5

Solution 6

It is useful to use multipliers $u = (1/3, 1/2)$. With this, we will obtain a valid inequality that includes only x_1 and has a fractional RHS that we will be able to round down. [Note: we don't present a general procedure here to get this to work, but see Gomory's method below!].

We obtain:

$$(7/6)x_1 + 0x_2 \leq 11/6$$

We can round down the coefficient of x_1 to the nearest integer to obtain the following valid inequality:

$$x_1 \leq 11/6$$

Now, as the LHS is integral for all points of X, we can reduce the RHS to the nearest integer, and we obtain the valid inequality for X:

$$x_1 \leq 1$$

This cuts off $x^* = (1.1, 0.4)$.

End Solution 6

Solution 7

Since there is only a single constraint, we can only find a single C-G valid inequalities, which results from writing the equality in the tableau as an inequality and taking the floor of the coefficients of the constraints:

$$x_1 + x_2 + x_3 - x_4 \leq 15$$

By adding a new slack variable the new constraint becomes:

$$x_1 + x_2 + x_3 - x_4 + x_5 = 15$$

This leads to the following simplex tableau: $\begin{bmatrix} vars & x_1 & x_2 & x_3 & x_4 & x_5 & RHS \\ z & 0 & 0.25 & 13.25 & 0.25 & 0 & 79.75 \\ x_1 & 1 & 1.75 & 1.95 & -0.05 & 0 & 15.95 \\ x_5 & 1 & 1 & 1 & -1 & 1 & 15 \end{bmatrix}$

Using one pivot to make x_1 basic, we end up with:

$$\begin{bmatrix} vars & x_1 & x_2 & x_3 & x_4 & x_5 & RHS \\ z & 0 & 0.25 & 13.25 & 0.25 & 0 & 79.75 \\ x_1 & 1 & 1.75 & 1.95 & -0.05 & 0 & 15.95 \\ x_5 & 0 & -0.75 & -0.95 & -0.95 & 1 & -0.95 \end{bmatrix}$$

A quick ratio test reveals that x_4 enters and x_5 leaves, which gives leads to the following tableau (rounded numbers):

$$\begin{bmatrix} vars & x_1 & x_2 & x_3 & x_4 & x_5 & RHS \\ z & 0 & 0.0526 & 13 & 0 & 0.263 & 80.0 \\ x_1 & 1 & 1.789 & 2.0 & 0.0 & -0.0526 & 16.0 \\ x_4 & 0 & -0.789 & 1.0 & 1.0 & -1.0526 & 1.0 \end{bmatrix}$$

All values on the right hand side are positive, no negative reduced costs can be found in the objective and the solution is integral. Hence $(16, 0, 0)$ this is the optimal solution to our original IP.

End Solution 7

Solution 8

Picking the first row of the tableau, which has a noninteger right hand side we have:

$$
\begin{array}{c|ccccccc}
\overline{a}_{1j} & 1.3 & -0.3 & 0 & -1 & 0 & 1 & 5.3 \\
\lfloor \overline{a}_{1j} \rfloor & 1 & -1 & 0 & -1 & 0 & 1 & 5.0 \\
\hline
\overline{a}_{1j} - \lfloor \overline{a}_{1j} \rfloor & 0.3 & 0.7 & 0 & 0 & 0 & 0 & 0.3
\end{array}
$$

Thus the associated Gomory cut would be

$$0.3x_1 + 0.7x_2 \geq 0.3$$

Using the same procedure for the other noninteger constraints, we find the remaining possible Gomory cuts:

$$
\begin{aligned}
0.1x_2 + 0.4x_4 &\geq 0.6 \\
0.2x_1 + 0.8x_2 + 0.5x_4 &\geq 0.7
\end{aligned}
$$

End Solution 8

Solution 9

By adding the third Gomory cut, we arrive at the following tableau (with an additional slack variable x_7):

$$\begin{bmatrix} vars & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & RHS \\ z & 0.1 & 0.3 & 0 & 0.2 & 0 & 0 & 0 & 23.1 \\ x_6 & 1.3 & -0.3 & 0 & -1.0 & 0 & 1 & 0 & 5.3 \\ x_3 & 0 & 1.1 & 1 & 0.4 & 0 & 0 & 0 & 1.6 \\ x_5 & -0.8 & -0.2 & 0 & -0.5 & 1 & 0 & 0 & 3.7 \\ x_7 & -0.2 & -0.8 & 0 & -0.5 & 0 & 0 & 1 & -0.7 \end{bmatrix}$$

Let's perform a dual pivot. Let x_7 leave the basis and x_2 enter:

$$\begin{bmatrix} vars & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & RHS \\ z & 0.025 & 0 & 0 & 0.0125 & 0 & 0 & 0.375 & 22.8375 \\ x_6 & 1.375 & 0 & 0 & -0.85 & 0 & 1 & -0.375 & 5.5625 \\ x_3 & -0.275 & 0 & 1 & -0.15 & 0 & 0 & 1.375 & 0.6375 \\ x_5 & -0.75 & 0 & 0 & -0.4 & 1 & 0 & -0.25 & 3.875 \\ x_2 & 0.25 & 1 & 0 & 0.625 & 0 & 0 & -1.25 & 0.875 \end{bmatrix}$$

This solution is optimal. Nonetheless, we haven't arrived at an integer solution yet. Additional Gomory cuts will be necessary.

End Solution 9