Problem Set 6
More IP: Branching and Bounding, Formulating, Tightening

AM121/ES121 — Fall 2019

Due 5 PM, Friday, November 8, 2019

Announcements

- The assignment is due by 5 PM, Friday, November 8, 2019.
- Please join a group. You may work with another student on this assignment and submit one writeup, but you must work together on every problem and state that you did this on your submission. It is ok to divide the writing up of the solutions, but not solving the problems.
- Readings: Jensen and Bard, sections 8.1–8.3.

Goals

- Practice solving integer programs via Branch and Bound.
- Gain a better understanding of formulation strength and its importance.
- Know how to apply your knowledge of integer programming and modeling to scheduling problems.
- Practice generating valid inequalities.
- Know how to solve an IP using the cutting planes method.

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1 Warm-up

Given two formulations for an integer program, a formulation P_1 is *stronger* than some P_2 if $P_1 \subset P_2$.

Task 1

Answer the following questions in your own words.

1. Why do we prefer stronger formulations?
2. What's the best possible formulation? Justify your answer.
3. Why might it be difficult to arrive at the best possible formulation?
4. Given binary variables $x_1, \ldots, x_m$ and continuous variable y, consider the constraint (C1):

$$\sum_{i=1}^{m} x_i \leq my$$

and the constraints (C2):

$$x_i \leq y \quad \forall i$$

Is there a sense in which one constraint is better than another? Give an argument based on the idea of stronger formulations, and demonstrate how all else equal, a formulation with a particular constraint is better than the alternative. Assume $m \geq 2$.

End Task 1

2 Branch and Bound

(Adapted from Wolsey, Chapter 7, Exercise 4.) Consider the following integer knapsack problem[1]:

$$\begin{array}{rl} \text{maximize} & 5x_1 + 8x_2 + 6x_3 + 2x_4 \\ \text{subject to} & 3x_1 + 6x_2 + 4x_3 + 2x_4 \leq 10 \\ & x \in Z_+^4 \end{array}$$

We can solve the LP relaxation using the Simplex method. Adding slack variable x_5 and equating z to the objective, we arrive at the following initial tableau:

$$\begin{bmatrix} var & z & x_1 & x_2 & x_3 & x_4 & x_5 & RHS \\ z & 1 & -5 & -8 & -6 & -2 & 0 & 0 \\ x_5 & 0 & 3 & 6 & 4 & 2 & 1 & 10 \end{bmatrix}$$

We have $x = (0, 0, 0, 0, 10)$ and $z = 0$. Following the smallest subscript rule, we let x_1 enter and leave on x_5:

$$\begin{bmatrix} var & z & x_1 & x_2 & x_3 & x_4 & x_5 & RHS \\ z & 1 & 0 & 2 & 2/3 & 4/3 & 5/3 & 50/3 \\ x_1 & 0 & 1 & 2 & 4/3 & 2/3 & 1/3 & 10/3 \end{bmatrix}$$

We have $x = (\frac{10}{3}, 0, 0, 0, 0)$ and $z = 50/3$. The reduced costs are non-negative and the solution is optimal for the LP relaxation.

[1] As opposed to 0-1 knapsack - where all decision variables are binary - here variables are integral.

Task 2

1. What does the LP relaxation tell us about the optimal objective value of the given integer knapsack problem?
2. Consider rounding the LP relaxation solution of the integer knapsack problem. What happens when you round up? What happens when you round down?
3. Consider using branch and bound to solve the integer knapsack problem. Since the value of x_1 in the optimal tableau of the LP relaxation is non-integral, we branch on $x_1 \geq 4$ and $x_1 \leq 3$. For now, consider the $x_1 \geq 4$ branch. Add this constraint (as an equality constraint with its associating slack variable) to the optimal tableau of the LP relaxation. Bring the tableau into dual-feasible form, and perform one step of the dual-simplex method. What do you notice?
4. Solve the integer knapsack problem above by branch and bound. Draw the branch and bound tree, annotating nodes and branches as we did in lecture. Be sure to number nodes in the order that they are expanded. Do not expand more nodes than necessary (Hint: you may use the fact that all objective coefficients are integral).

   Use the following rules:

   - Expand $\geq$ branches before $\leq$ branches.
   - Expand in a depth-first fashion.
   - If more than one variable is fractional, expand on the variable with the smallest subscript.

   We recommend using AMPL to solve subproblems, and we've posted the AMPL code for the basic linear program on the website. If you enjoy doing extra work, feel free to use Dual-Simplex and solve by hand (or with the help of Maple, say). For submission purposes, you need only include the completely annotated branch and bound tree.

End Task 2

3 R.O.B. does your job

The School of Engineering and Applied Sciences have just purchased a robot (R.O.B.) capable of doing a variety of jobs. Hearing this news, students and faculty submit n jobs for R.O.B., where job i takes time $t_i \geq 0$ to process. Since R.O.B. can only process one job at a time, he wishes to schedule the jobs to minimize the average completion time for the jobs.For example, consider five jobs with processing times:

j	1	2	3	4	5
t_j (seconds)	4	20	46	36	6

If R.O.B. chose the schedule order 1,5,2,4,3, then the completion times would be 4,10,30,66,112, giving an average completion time of $\frac{222}{5}$.

Task 3

1. Formulate a general mathematical model of the problem as a mixed integer program. Be sure to specify the type (binary, integer, or continuous) of each variable and describe the elements of your model. Chances are, you will need to use a big-M in your formulation. Remember that your formulation should be general and not specific to the example. No AMPL implementation is necessary. (Hint: you may find introducing binary variables y_{ij} to indicate whether or not job i is scheduled before or after job j for all $i \neq j$ useful for your formulation, as well as variables c_i indicating the completion time of job i.)

2. If you have used a big-M in your formulation, explain how the constant should be set to improve solve time. Then, derive an expression that can be used to set the constant for any instance of the problem. If you did not use a big-M above, please feel free to skip this part.

3. The department thanks you for your work. They wonder if you can modify your formulation to take 'precedence' constraints into account. For example, a 'precedence pair' $(1, 3)$ means that job 1 must finish before job 3 starts. Update your formulation to take precedence constraints into account. Describe any changes to your model and any assumptions made. Your answer should be brief!

4. The department wishes to distinguish between student jobs and faculty jobs, such that if the average completion time on faculty jobs is less than 70% of the average completion time on student jobs, then the average completion time of student jobs must be no greater than 30% above the average completion time to avoid student complaints. Update your formulation to take these constraints into account. Describe any changes to your model and argue for its correctness. If using big-M constants, derive expressions on how to set the constants tightly. You may assume that you can index over the set of student jobs or faculty jobs.

5. (For those who want a challenge! Here's a completely optional problem for extra credit.) The department wishes to distinguish between student jobs and faculty jobs, such that if the average completion time on faculty jobs is less than the average completion time on student jobs, then a third of student jobs must complete before the average completion time to avoid student complaints.

   Can you model this logical statement with an IP? If so, update your formulation to take these constraints into account. Describe any changes to your model and argue for its correctness. If using big-M constants, derive expressions on how to set the constants tightly. If the logical statement cannot be modeled with an IP, explain why not.

End Task 3

4 Inequalities

Task 4

For each of the examples below a set X and a point x or (x, y) are given. Find a valid inequality for X cutting off the point. Describe briefly how you came up with the inequality and why the inequality is valid. (Hint: A C-G inequality can be used for part 2.)

1.
$$X = \{(x, y) \in R^1_+ \times Z^1_+ : x \leq 11, x \leq 5y\}$$
$$(x, y) = (11, \frac{11}{5})$$

2.
$$X = \{x \in Z^4_+ : 5x_1 + 10x_2 + 6x_3 + 2x_4 \leq 32\}$$
$$x = (0, 0, \frac{16}{3}, 0)$$

End Task 4

5 Covers

Task 5

In each of these examples below a set X and a point x are given. Find a valid inequality for X cutting off x. Note that $B = \{0, 1\}$ (Hint: For part 2, find a cover inequality and then strengthen it).

1.
$$X = \{x \in B^5 : 7x_1 + 5x_2 + 5x_3 + 4x_4 + 2x_5 \leq 13\}$$
$$x = (\frac{1}{7}, 1, \frac{4}{5}, \frac{1}{4}, 1)$$

2.
$$X = \{x \in B^5 : 9x_1 + 8x_2 + 6x_3 + 6x_4 + 5x_5 \leq 14$$
$$x = (\frac{1}{4}, \frac{1}{8}, \frac{3}{4}, \frac{3}{4}, 0)$$

End Task 5

6 Cutting planes

Consider the following integer program:

$$\begin{aligned} \max \quad & -2x_1 + 4x_2 \\ s.t. \quad & 2x_1 + x_2 \leq 5 \\ & -4x_1 + 4x_2 \leq 5 \\ & x_i \in Z_+^2 \end{aligned}$$

Adding slack variables x_3 and x_4 and applying the Simplex method, we arrive at the following optimal tableau:

$$\begin{bmatrix} var & z & x_1 & x_2 & x_3 & x_4 & RHS \\ z & 1 & 0 & 0 & 2/3 & 5/6 & 15/2 \\ x_1 & 0 & 1 & 0 & 1/3 & -1/12 & 5/4 \\ x_2 & 0 & 0 & 1 & 1/3 & 1/6 & 5/2 \end{bmatrix}$$

Task 6

For the following tasks, please feel free to use Maple or other mathematical software to your advantage as you see fit.

1. Graph the feasible region of the LP relaxation.
2. Generate a Gomory cut for the second constraint (the one with RHS = 5/2).
3. Write the cut in terms of x_1 and x_2.
4. Update the feasible region in your graph to include the reformulated cut you generated. Label this cut as '1st cut'.
5. Update the optimal tableau to include this cut. Bring the tableau into dual-feasible form.
6. Use dual simplex to find the new optimal tableau.
7. Generate the Gomory cut for the constraint with a RHS of 3/4 in the optimal tableau to the previous part.
8. Write this cut in terms of the variables x_1 and x_2.
9. Update the feasible region in your drawing to include this cut. Label it 'Second Cut'.
10. Update the optimal tableau to add this cut.
11. Apply dual simplex to generate an optimal tableau.
12. Determine the optimal IP solution and objective value from this optimal tableau.

End Task 6

7 We like branch and bound, but...

We like branch and bound, we really do. Often, the technique allows us to prune away large portions of the search tree and when applied with 'good' heuristics can lead to quick solve times of IPs. However, this is not always the case.

Task 7

Consider the following Binary Integer Program:

$$\begin{aligned} \min \quad & u_{n+1} \\ s.t. \quad & 2u_1 + 2u_2 + \ldots + 2u_n + u_{n+1} = n \\ & u_i \in \{0, 1\} \end{aligned}$$

Assume that n is odd. Show that any branch and bound algorithm that branches by setting fractional variables to either zero or one and uses LP relaxations to compute bounds will require the enumeration of an exponential number of subproblems. Note:

1. a formal proof is not required nor expected for full credit, but may be attempted for extra credit.
2. this question will not be worth a lot of points, but you should attempt it nevertheless.

End Task 7

8 Turning in your assignment (please follow the guidelines)

Final Task 8

Turn in your assignment by 5 PM, Friday, November 8, 2019. Scan your homework if you completed it on paper and make sure the PDF is eligible. If you worked in pairs, please create a group on Gradescope and make ONE submission per pair.

For AMPL exercises, this includes any model and data files you have created. When writing down the solution from AMPL, always include both the objective value and the values assigned to variables (when the program is feasible and bounded.) Scan the AMPL model and data files you have created for this assignment, and compile them into one PDF for submission. Please merge PDFs instead of submitting multiple PDFs.

End Task 8

Congratulations on completing your sixth AM/ES 121 assignment!