AM 121 /ES 121 Introduction to Optimization: Models and Methods
Practice Midterm 2

Prof. David C. Parkes

Fall 2016

Here are some practice questions to help to prepare for the midterm. The midterm will not contain as many questions.

1. MIP modeling

The Boston Harbor Condo Project will contain both homes and apartments. The site can accommodate up to 10,000 such units (homes or apartments) in total. The project must also contain either a swimming-tennis complex (cost = \$28 million) or a sailboat marina (cost = \$12 million), but not both. If a marina is built then the number of homes must be at least triple the number of apartments. Each apartment will yield profit of \$80,000 and each house \$60,000. Formulate an IP to help Boston Harbor maximize profits.

2. MIP modeling

WIP Publishing sells textbooks to college students. WIP has two sales reps available to assign to the A-G state area (see Fig. 1). The number of college students (in thousands) in each state is given in the figure. Each rep must be assigned to two adjacent states (e.g., A and B is OK but not A and D). WIP's goal is to maximize the number of total students in the states assigned to the reps. More than one rep can be assigned to the same state but the students in that state should not be counted twice! Formulate an IP whose solution will tell you where to assign the sales reps.

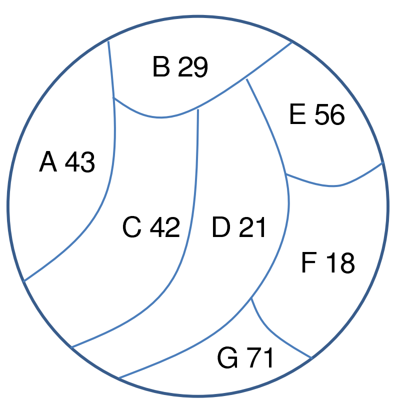

Figure 1: Diagram of the state area

3. Branch and Bound

While running the Branch-and-Bound method we reach node 27 which still needs to be processed. The subproblem at this node is the following *minimization* problem, with coefficients c_1, c_2, a_1 and a_2 left unspecified:

$$\begin{aligned} \min_{x_6, x_7} \quad & c_1 x_6 + c_2 x_7 + 15 \\ \text{s.t.} \quad & a_1 x_6 + a_2 x_7 \geq 15 \\ & x_6, x_7 \in \{0, 1\} \end{aligned}$$

The current incumbent feasible solution has an objective value of 34.

For each of the following, provide *example values* for parameters c_1, c_2, a_1 and a_2 such that the condition is satisfied. You should assume that the parameters take only integer values. The question asks about the LP relaxation of the IP corresponding to the subproblem at this node.

(a) Node 27 is fathomed due to infeasibility of the LP relaxation.

(b) Node 27 is fathomed where the solution to the LP relaxation is IP feasible and the incumbent does not change.

(c) Node 27 is fathomed where the solution to the LP relaxation is IP feasible and the incumbent is updated to the new objective value.

(d) Node 27 is fathomed where the solution to the LP relaxation is not IP feasible.

(e) Node 27 is not fathomed.

4. Quick fire: Integer programming

(a) True or False: For IP problems, the number of integer variables is generally more important in determining the computational difficulties than the number of constraints.

(b) True or False: The feasible region for the LP relaxation of an IP is always a subset of the feasible region for the IP problem.

(c) True or False: Branch-and-cut uses cutting planes in place of the solution to LP relaxations to solve MIPs.

(d) True or False: Strong branching can be considered to do a little bit of look ahead in the search tree before deciding which open subproblem to solve next.

(e) True or False: Any integer program can be formulated as a linear program for which all extremal solutions are feasible, integer solutions.

(f) True or False: Either depth-first search or best-bound search is used for node selection in Branch-and-bound search.

(g) True or False: Gomory's algorithm is complete for integer programming (i.e., it can solve any IP).

(h) True or False: A common method for variable selection is to branch on the most fractional variable.

5. Conceptual: Integer programming

(a) Consider the following optimal tableau for an LP relaxation of an IP that is a maximization problem and where all variables $\{x_1, x_2, x_3, x_4\}$ are required to be integer.

$$\begin{array}{lllll} z & & & +\frac{56}{11}x_3 & +\frac{30}{11}x_4 & = 126 \\ & x_1 & & -\frac{1}{22}x_3 & +\frac{3}{22}x_4 & = \frac{9}{2} \\ & & x_2 & +\frac{7}{22}x_3 & +\frac{1}{22}x_4 & = \frac{7}{2} \end{array}$$

Generate a Gomory cut for this tableau. Demonstrate how to extend the current tableau in order to incorporate this cut. (*There is no need to solve the new tableau.*)

(b) Formulate as an IP the requirement that at least two of the following four inequalities must hold:

$$\begin{aligned} 5x_1 + 3x_2 + 3x_3 - x_4 &\le 10 \\ 2x_1 + 5x_2 - x_3 + 3x_4 &\le 10 \\ -x_1 + 3x_2 + 5x_3 + 3x_4 &\le 10 \\ 3x_1 - x_2 + 3x_3 + 5x_4 &\le 10 \end{aligned}$$

(c) Consider a binary integer program (all variables are $\{0, 1\}$) with constraint:

$$x_1 + 3x_2 + 2x_3 + 4x_4 \le 5$$

Identify all the minimal set covers for this constraint, and provide the corresponding cover inequalities.

(d) Formulate the following as an MIP: $\max c^T x$ where $x \in X \setminus \tilde{x}$ (i.e., x should not take the value $\tilde{x}$), $X = \{x \in \{0,1\}^n : Ax \le b\}$ and $\tilde{x} \in X$

6. Valid inequalities and Cuts

Consider the following IP:

$$\begin{aligned} \max \quad & 2x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 5 \\ & -x_1 + x_2 \le 2 \\ & x_1 - x_2 \le 2 \\ & x_1 + x_2 \ge 3 \\ & x_1, x_2 \ge 0, \text{and integer} \end{aligned}$$

The solution to the LP relaxation of this IP is $x^* = (x_1, x_2) = (1.5, 3.5)$. Determine which of the following inequalities are valid. Furthermore, determine which of the inequalities separate x^*.

(a) $x_1 \leq 3$

(b) $x_2 \leq 4$

(c) $x_1 + 3x_2 \leq 10$

[Hint: for one of these you will find it useful to draw a picture.]

7. Valid inequalities and Cuts

(a) Consider the set $X = \{(x, y) : x \leq 12y, 0 \leq x \leq 15, y \in \{0, 1, 2, \ldots\}\}$. Identify an inequality that when added to polytope $P = \{x \leq 12y, 0 \leq x \leq 15, y \geq 0\}$ provides the convex hull of X. (Hint: a sketch of X might help.)

(b) Consider $X = \{x \in \{0, 1\}^6 : 13x_1 + 8x_2 + 5x_3 + 5x_4 + 4x_5 + x_6 \leq 21\}$. Find *two* minimal cover inequalities, including one that can be extended. Give the extended cover inequality. (There is only one cover inequality that can be extended.)

8. Formulation strength

A website agrees to provide s_i user impressions to each of two advertisers $i = 1, 2$. The site receives a payment w_i per impression, but incurs a penalty c_i if the total demand s_i is not met.

Decision variables x_i denote the number of impressions allocated to each advertiser.

(a) Adopt "big-M"s to formulate this problem as a MIP, introducing terms into the objective and additional constraints as necessary. (Set X captures the rest of the feasibility information about the problem.)

(b) By assuming either an upper or lower bound on the number of impressions that will be allocated to each advertiser, can you express tighter big-M's?

9. Markov chains

Archy Leach, a recent graduate of Bam Bam Dentistry College, has opened his first office. The office has one dentist's chair and a waiting area that seats two. The total capacity is three. In any given time period, exactly one of three things may happen:

- A patient may arrive at the office.
- A patient may leave the office.
- There are no arrivals or departures

The probability that a patient arrives is 0.01. But if the patient arrives and finds the waiting area full then the patient cannot stay in the office. When there are one or more patients in the office, then one patient is being treated and the probability that a treated patient will depart is 0.02.

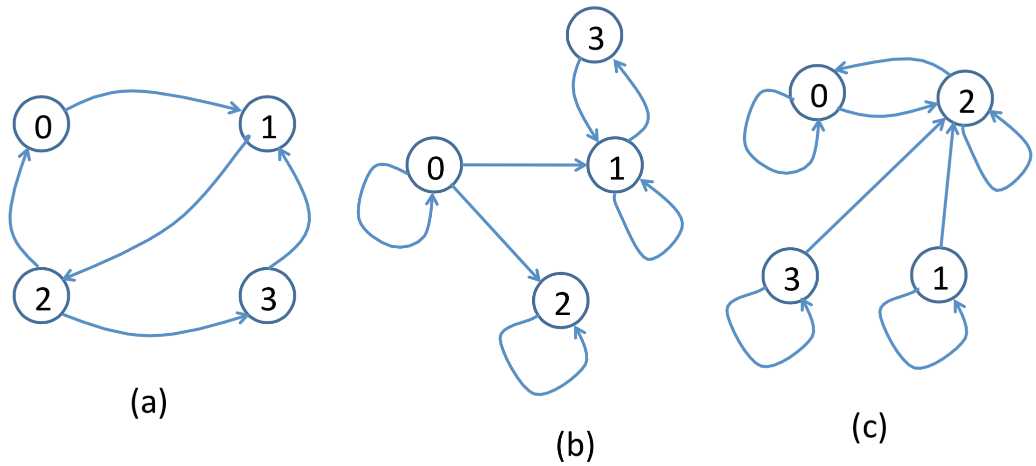

Figure 2: Three Markov chains (a), (b) and (c)

(a) Define the Markov chain to model this problem, including the states and the transition matrix **P**.

(b) Compute the steady state probabilities and use them to answer the following questions:

(i) what percent of time would you expect the office to be full?

(ii) what percent of time would you expect the office to be empty?

(iii) if Archy earns $50 each time a treated patient leaves the office, what are his expected daily earnings (assume 500 time periods in a day).

10. Markov Chains

For each of the three Markov chains shown in Fig.2, classify the states, and label each class as recurrent or transient, and as periodic or aperiodic. If a class is periodic give its period.

11. Quick fire: Markov chains and MDPs

(a) True or False: The LP formulation of the optimal MDP policy under the limit-average reward criterion will lead to deterministic policies.

(b) True or False: The limit of the n-step transition probability $p_{ij}^{(n)}$ as $n \to \infty$ in a Markov chain exists when the Markov chain is irreducible and aperiodic.

(c) What do the variables $\pi(s, a)$ represent in the LP formulation of the optimal MDP policy under the limit-average reward criterion?

12. Markov Decision Processes

Alex lives in a place that may be either rainy, cloudy or dry. The weather changes in uncertain ways from day to day (but in a way that is independent of the weather on previous days.)

Alex can also affect the weather! She likes to dance when it is cloudy, but dancing makes it likely it will rain the next day (irrespective of the weather today). When it is raining she needs to use an umbrella to be happy (and unlike Fred Astaire, she can't dance at the same time as carrying her umbrella.) When it is sunny her favorite thing to do is to relax.

Formulate an illustrative, concrete Markov decision process (i.e., provide example numbers for transitions and rewards) for Alex's decision problem. You should define the (a) states, (b) actions, (c) reward for each action in each state, and (d) transition from state to next state for each action. You should also suggest the optimality criterion. *You do not need to give the LP formulation, or solve for an optimal policy!*