AM 121: Intro to Optimization Models and Methods

Fall 2016

Lecture 20: Stochastic Optimization

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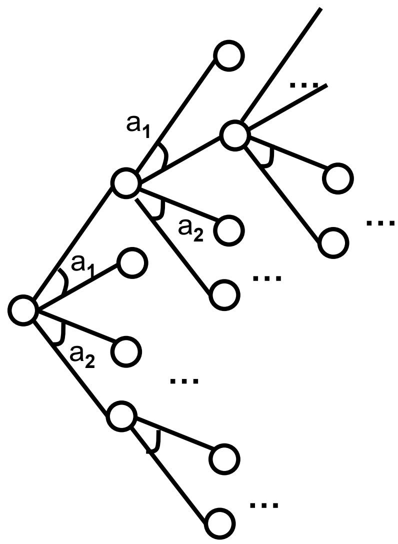
Lesson Plan

- Two-stage stochastic optimization
  - Stage one; stage two (recourse)
- Example: the Farmer's problem, the Contractor's problem
- Optimal stochastic solution
- EVPI and VSS
- Analytic solution method, Sample Average Approximation method

Reading: "A tutorial on stochastic programming," Schapiro and Philpott, March 2007 (sections 1 and 2)

Stochastic Optimization



MDP: M=(S,A,P,R)
m states, n actions

Decision variables in the LP are π(s,a). Can only solve if m*n is small.

But n may be large.

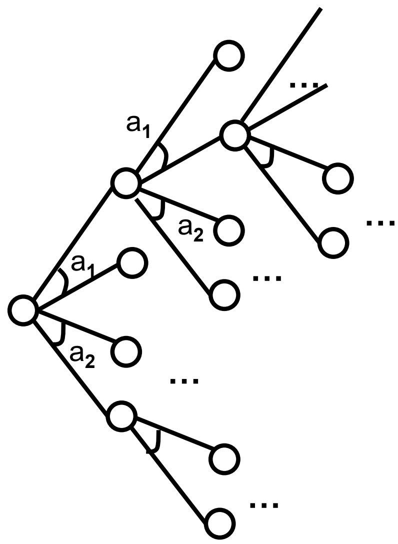
Two-stage stochastic optimization is special case for two periods.

Example: Contractor's problem

- Accept some projects at time 1. Gain revenue.
- #actions = # subsets of projects
- Exponentially large!

- At time 2, know the labor needed to complete each project. Can recruit additional workers.

Stochastic Optimization

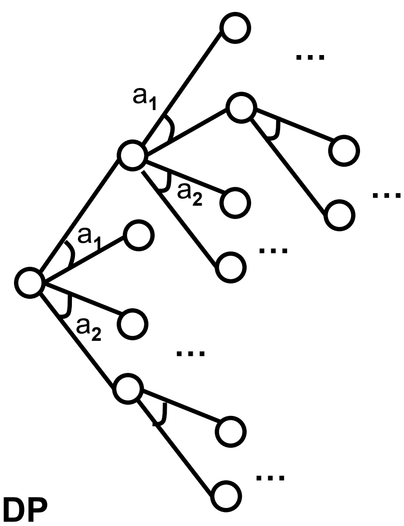


MDP: M=(S,A,P,R)
m states, n actions

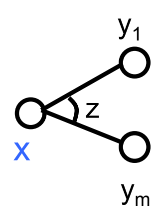
Decision variables in the LP are π(s,a). Can only solve if m*n is small.

But n may be large (e.g., subsets of projects.)

m may also be large (e.g. all things that can go wrong with a project!)



MDP
(state transition depends on action)



Two-stage stochastic optimization:

Decision $x \in X$ at time step 1

Random event z: models *all* uncertainty (e.g., what could go wrong with every possible set of projects!)

Decision y at time step 2.

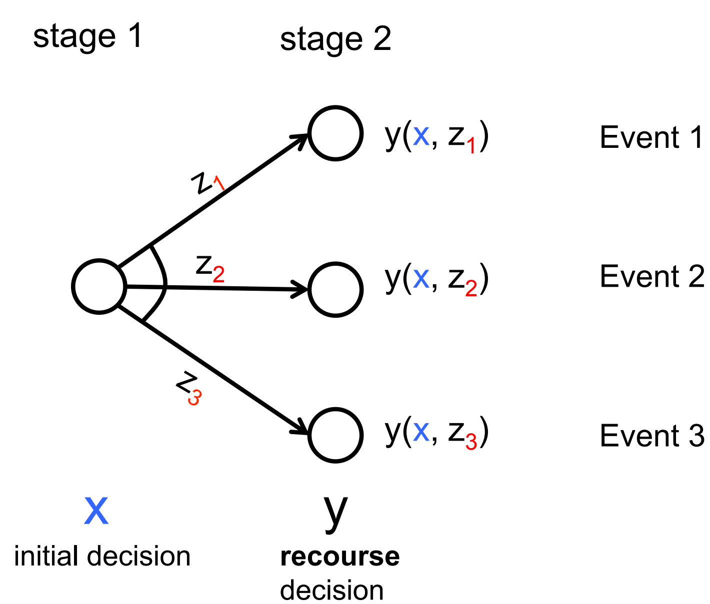
Examples

- **T-shirt problem.** Buy x t-shirts at time 1, cost c. At time 2, uncertain demand z realized and need to either back-order or hold excess inventory.

- **Farmer's problem**. Plant x crops at time 1, incur cost. At time 2, uncertain weather z realized. Buy/sell crops to ensure enough feed for animals, maximize profit.

- **Contractor's problem**. Accept projects x at time 1, get payments. At time 2, realize amount of labor z needed for each project, recruit workers as necessary.

Two-stage stochastic optimization



Example: Farmer's problem

(Birge & Louveaux' 97)

- A farmer has 500 acres to plant wheat, grain and sugar beets. Needs min amount of wheat/corn to feed animals. Can also buy/sell in period 2.
- Without uncertainty about yield:

	Wheat	Corn	Beets
Yield (T/acre)	2.5	3	20
Cost ($/acre)	150	230	260
Sell price ($/T)	170	150	36 (under 6000 T) 10 (above 6000 T)
Purchase price ($/T)	238	210	-
Min requirement (T)	200	240	-

Formulation without Uncertainty

- Stage 1: x_1, x_2, x_3 = acres to plant for each crop
- Stage 2:
  - y_1^b, y_1^s = amount to buy, sell of crop 1 (w)
  - y_2^b, y_2^s = amount to buy, sell of crop 2 (c)
  - y_3^{s1}, y_3^{s2} = amount to sell of crop 3 at high, low price
- max $-150x_1-230x_2-260x_3$ $- 238y_1^b +170y_1^s - 210y_2^b + 150y_2^s + 36y_3^{s1} + 10y_3^{s2}$

s.t. $x_1 + x_2 + x_3 \leq 500$ (land)

$2.5x_1 + y_1^b - y_1^s \geq 200$
$3x_2 + y_2^b - y_2^s \geq 240$ } (quotas)

$y_3^{s1} + y_3^{s2} \leq 20x_3$
$y_3^{s1} \leq 6000$ } (beets)

$x_1, x_2, \ldots, y_1^b, \ldots y_3^{s2} \geq 0$

Optimal solution (No uncertainty)

	Wheat	Corn	Beets
acres	**120**	**80**	**300**
yield (T)	300	240	6000
sales	100	-	6000
purchase	-	-	-

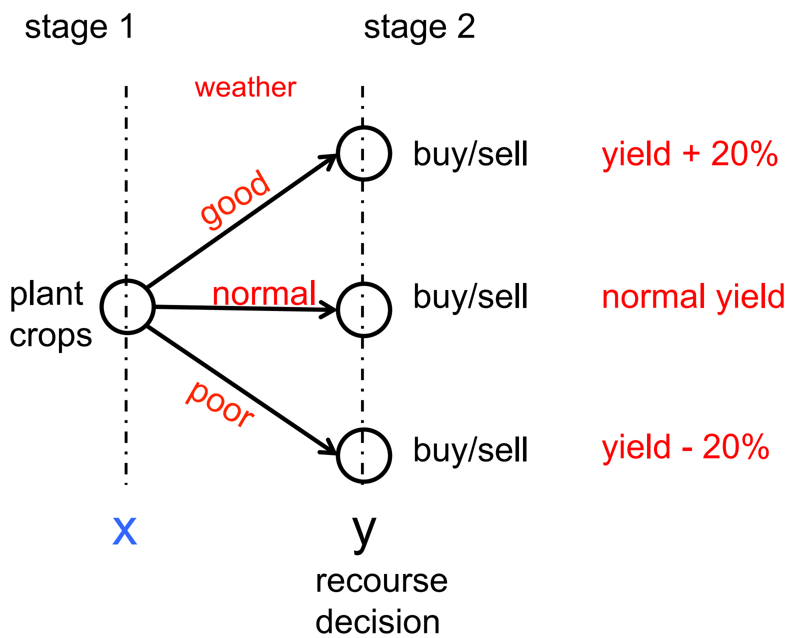
High profit on sales

High profit up to 6000T

Feed animals

Introducing uncertainty

- Weather may be "good", "normal" or "poor." This affects the yield on each crop
- Given this uncertainty, what is optimal decision in stage one about which crops to plant?



Warm-up: An omniscient farmer

Suppose the farmer can predict the weather…

Optimal decision if weather good

	Wheat	Corn	Beets
acres	**183.3**	**66.7**	**250**
yield (T)	550	240	6000
sales	350	-	6000
purchase	-	-	-

Optimal decision if weather normal

	Wheat	Corn	Beets
acres	**120**	**80**	**300**
yield (T)	300	240	6000
sales	100	-	6000
purchase	-	-	-

Optimal decision if weather poor

	Wheat	Corn	Beets
acres	**100**	**25**	**375**
yield (T)	200	60	6000
sales	-	-	6000
purchase	-	180	-

Comparing the Omniscient solutions

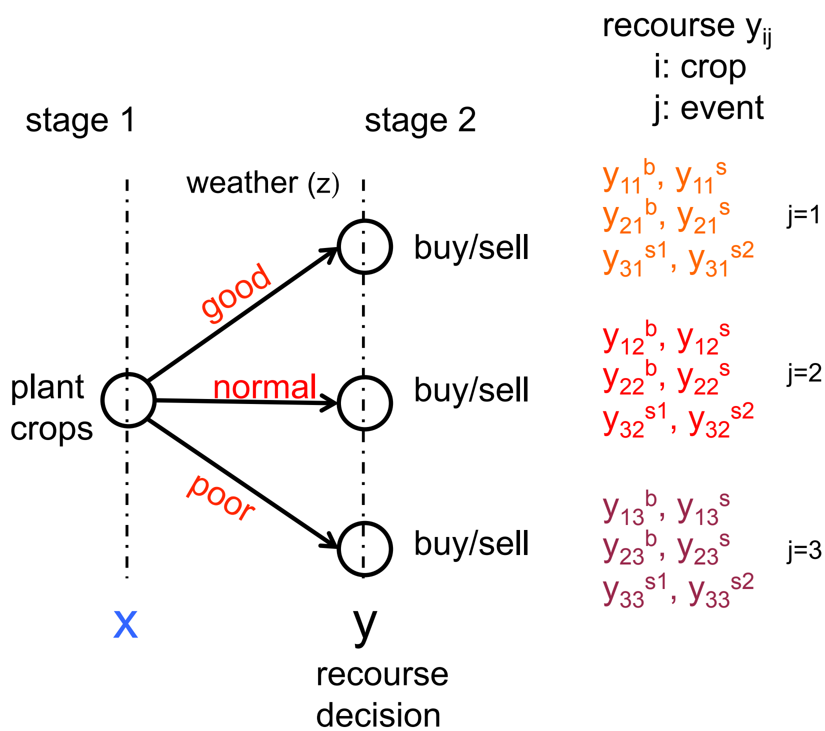
	good	normal	poor
wheat (acres)	183.3	120	100
corn (acres)	66.7	80	25
beets (acres)	250	300	375

A lot of variation: best to plant between 183 and 100 acres of wheat, depending on the weather.

What to do?!

Two-stage Stochastic Optimization

- Uncertain event. Weather is {good, normal, poor}
- Probability 1/3 on each
- What to do in period one, and in period two (recourse)?



- max $-150x_1 - 230x_2 - 260x_3$
  $+1/3\,(-238y_{11}^b + 170y_{11}^s - 210y_{21}^b + 150y_{21}^s + 36y_{31}^{s1} + 10y_{31}^{s2})$
  $+1/3\,(-238y_{12}^b + 170y_{12}^s - 210y_{22}^b + 150y_{22}^s + 36y_{32}^{s1} + 10y_{32}^{s2})$
  $+1/3\,(-238y_{13}^b + 170y_{13}^s - 210y_{23}^b + 150y_{23}^s + 36y_{33}^{s1} + 10y_{33}^{s2})$

s.t. $x_1 + x_2 + x_3 \le 500$ — stage 1

$$\begin{aligned} 3x_1 + y_{11}^b - y_{11}^s &\ge 200 \\ 3.6x_2 + y_{21}^b - y_{21}^s &\ge 240 \\ y_{31}^{s1} + y_{31}^{s2} &\le 24x_3 \\ y_{31}^{s1} &\le 6000 \end{aligned}$$

stage 2: good weather

$$\begin{aligned} 2.5x_1 + y_{12}^b - y_{12}^s &\ge 200 \\ 3x_2 + y_{22}^b - y_{22}^s &\ge 240 \\ y_{32}^{s1} + y_{32}^{s2} &\le 20x_3 \\ y_{32}^{s1} &\le 6000 \end{aligned}$$

stage 2: normal weather

$$\begin{aligned} 2x_1 + y_{13}^b - y_{13}^s &\ge 200 \\ 2.4x_2 + y_{23}^b - y_{23}^s &\ge 240 \\ y_{33}^{s1} + y_{33}^{s2} &\le 16x_3 \\ y_{33}^{s1} &\le 6000 \end{aligned}$$

stage 2: bad weather

$x_1, x_2, \ldots, y_{11}^b, \ldots y_{33}^{s2} \ge 0$

Optimal solution with Uncertainty

	Wheat	Corn	Beets
Acres	170	80	250
Yield (T)	510	288	6000
Sales (T)	310	48	6000
Purchase (T)	-	-	-
Yield (T)	425	240	5000
Sales (T)	225	-	5000
Purchase (T)	-	-	-
Yield (T)	340	192	4000
Sales (T)	140	-	4000
Purchase (T)	-	48	-

Profit: $108,390

"omniscient solution" (if knew weather) would bring expected value 1/3 (167+118+59) ≈ $114,667

	stochastic	good	normal	poor
wheat (acres)	170	183.3	120	100
corn (acres)	80	66.7	80	25
beets (acres)	250	250	300	375
profits	$108,390	$167,000	$118,000	$59,000

"**in expectation solution**"
(Assume yield = expected yield. Has **expected value** $107,240).

Expected value of Perfect Information (EVPI): $114,667 - $108,390 = $6,277

Value of Stochastic Solution (VSS): $108,390 - $107,240 = $1,150

EVPI and VSS

- $C(x)$: stage 1 value of stage 1 decision
- $Q(x,z)$: stage 2 value of opt recourse given (x,z)
- $V(z)$: total value of optimal stage 1 and 2 decisions given z

- EVPI: *expected value of perfect information*

$$\mathbf{E}_z[V(z)] - \max_x(C(x) + \mathbf{E}_z[Q(x,z)])$$

- VSS: *expected value of stochastic solution*

$$\max_x (C(x) + \mathbf{E}_z[Q(x,z)]) - (C(x^*)+\mathbf{E}_z[Q(x^*,z)])$$

  where x^* is opt "in expectation" stage 1 decision

Two-Stage Stochastic Programming

- First-stage decision $x \in R^n$. Event $z=(q,T,W,h)$ defines second-stage problem.
- $y \in R^m$ second-stage decision

$$\max_x c^T x + \mathbf{E}_z[Q(x,z)]$$
$$\text{s.t. } Ax \leq b$$
$$x \geq 0$$

First-stage problem: max the value of t=1 and **expected** value of t=2

where

$$Q(x,z) = \max_y q^T y$$
$$\text{s.t. } Tx + Wy \leq h$$
$$y \geq 0$$

Second-stage problem: given (x, z), maximize value of t=2.

Example: Farmer's problem

- Only T matrix is uncertain in the farmer's problem
- Let t_i denote **yield** of crop i; realized event $z=(t_1,t_2,t_3)$
- Second-stage decision problem:

$$
\begin{aligned}
Q(x, z) = \max\ & -238y_1^b + 170y_1^s - 210y_2^b + 150y_2^s + 36y_3^{s1} + 10y_3^{s2} \\
\text{s.t. } & t_1x_1 + y_1^b - y_1^s \geq 200 \\
& t_2x_2 + y_2^b - y_2^s \geq 240 \\
& y_3^{s1} + y_3^{s2} \leq t_3x_3 \\
& y_3^{s1} \leq 6000
\end{aligned}
$$

Maximize period 2 value given (x, z).

Computational approaches

(1) **Small number of possible events**, can form a single LP and solve (e.g., Farmer's problem.)

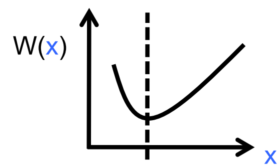
(2) Event z is a continuous r.v., but can solve for $\mathbf{E}_z[Q(x,z)]$ analytically and then solve for stage 1 analytically.

(3) Cannot use approach (1) or (2). Use **sample average approximation** (**SAA**).

Analytical approach: Example

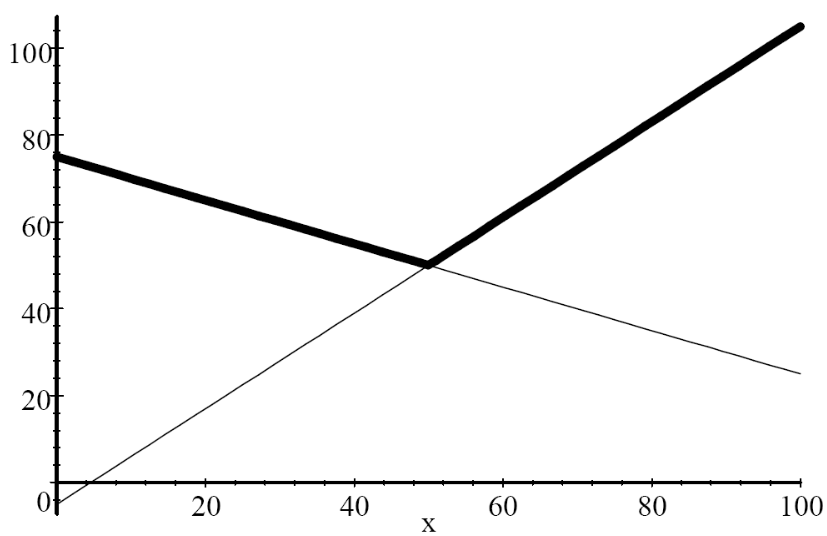
- We're giving away Harvard-Yale T-shirts!
- Must decide how many to order x at cost c=$1. Demand z~U(0,100) uncertain
- Cost b=$1.5 for "backorder cost" if z>x, holding cost h=$0.1 if x>z.

- min cx + $\mathbf{E}_z$[Q(x,z)]
- Q(x,z) = b max(z-x,0) + h max(x-z,0)



- Write $\mathbf{E}_z$[Q(x,z)] = W(x)
- Solve for x^* by first-order optimality: c+W'(x)=0

Example: cx + Q(x,z) for z=50



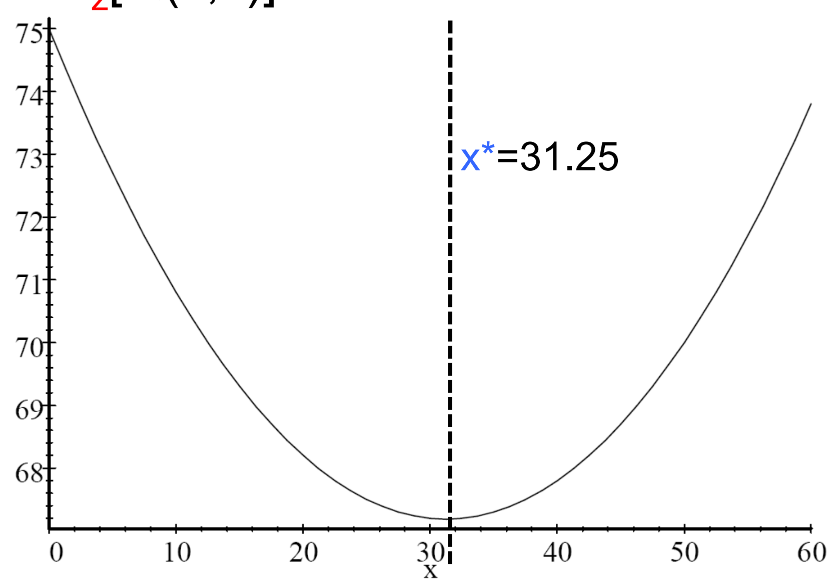
cx +Q(x,50) = 1x + 1.5 max(50-x,0)+ 0.1 max(x-50,0)

- $cx + \mathbf{E}_z[Q(x,z)] = b\mathbf{E}_z[z] + (c-b)x + (b+h)\int_0^x F(z)\,dz$

  $= (1.5)(50) + (-0.5)x + 1.6\int_0^x z/100\,dz$

  $= 75 - 0.5x + 0.008x^2$

$cx + E_z[Q(x,z)]$



Computational approaches

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Sample Average Approximation

(Kleywegt et al. 2001)

- **Approximate $\mathbf{E}_z[Q(x,z)]$** by sampling K possible events, with

$$\mathbf{E}_z[Q(x, z)] \approx (1/K) \sum_{k=1}^{K} Q(x, z_k)$$

- Solve:

$$\min_{x,\, y} \quad c^T x + (1/K) \sum_k q_k^T y_k$$
$$\text{s.t. } T_k\, x + W_k\, y_k \leq h_k \qquad \text{for all k}$$

Example: T-shirt problem

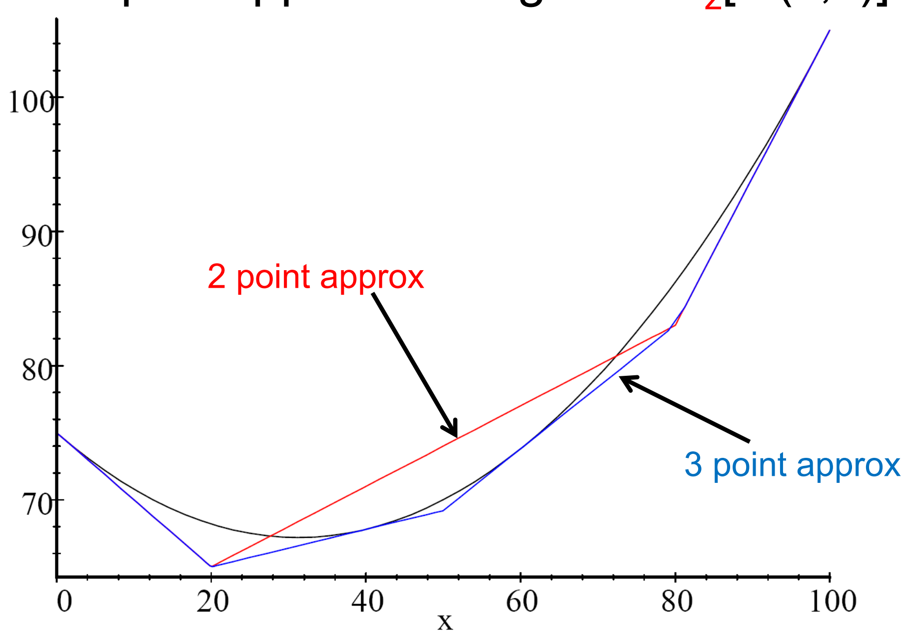
- Consider two realized (demand) events:

  $z_1 = 20$, $z_2 = 80$

- Objective: $cx + \mathbf{E}_z[Q(x,z)] = cx + ½Q(x,z_1) + ½Q(x,z_2)$
- Have one set of period 2 decision variables per sampled event

- Can also consider three (or more!) scenarios…

Example: Approximating $cx + E_z[Q(x,z)]$



Theoretical properties of SAA

(Kleywegt et al. 2001)

- V_K = exp. value of approx solution for K samples
- S_K^d = all solutions within distance d of approx soln
- V^* = exp value of optimal solution
- S^d = all solutions within distance d of optimal soln

Theorem:

(1) $\lim_{K \rightarrow \infty} V_K = V^*$.

(2) $S_K^d \subset S^d$ with probability 1 as $K \rightarrow \infty$, for any distance d.

Example Applications

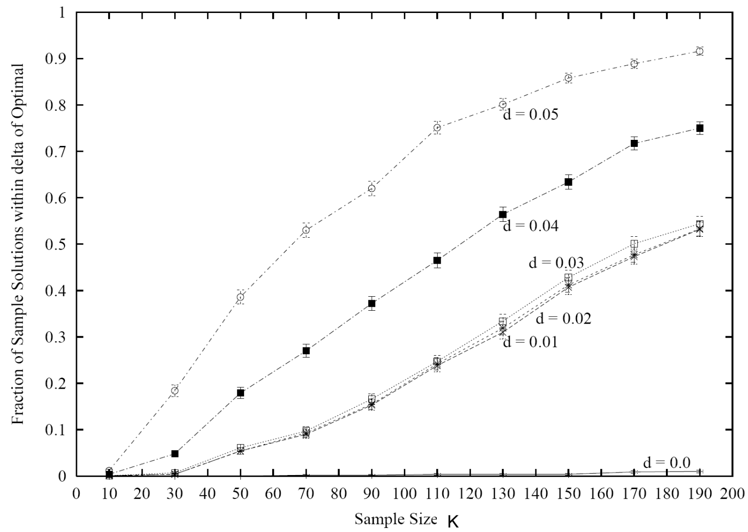
(Kleywegt et al. 01; van Hentenryck)

- Contractor's problem.
- Airline crew scheduling. Assign crew to routes, but each route takes an uncertain time. May need to pay over-time.
- Power restoration (schedule repair crews, schedule power restoration. Minimize overall size of blackout.)
- Scheduling evacuations (save as many lives as possible.)

Contractor's Problem

(Kleywegt et al. 01)

20 projects. Fraction of solutions within delta (d) of optimal:



Evacuation planning

(van Hentenryck 2015)

See additional slide deck.

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Summary: Stochastic Optimization

- Scales up MDPs to settings with many decisions, many possible states

- Solve first-stage in anticipation of second-stage (recourse).
- Find value-maximizing first-stage decision
- “Optimal stochastic solution”

- Sample average approximation method