AM 121: Intro to Optimization Models and Methods

EO 2: Paired Kidney Donation

David C. Parkes

Matching Problems

- Two-sided matching
- One-sided matching with strict preferences
- One-sided matching
I: Two-sided Matching

- Students to (public) schools; Medical interns to residencies; TAs to professors

- Agents in two sets; agent in one set has **strict** preferences over agents in other set

- A **matching**: each agent assigned to at most one agent on other side

Example (unstable!)

<table>
<thead>
<tr>
<th>the boys</th>
<th>and girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jake</td>
<td>Twiggy</td>
</tr>
<tr>
<td>Holly &gt; Claire &gt; Twiggy &gt; Jill</td>
<td>Jake &gt; Elwood &gt; Curtis &gt; Ray</td>
</tr>
<tr>
<td>Elwood</td>
<td>Claire</td>
</tr>
<tr>
<td>Claire &gt; Jill &gt; Twiggy &gt; Holly</td>
<td>Jake &gt; Curtis &gt; Elwood &gt; Ray</td>
</tr>
<tr>
<td>Curtis</td>
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</tr>
<tr>
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<td>Ray &gt; Curtis &gt; Elwood &gt; Jake</td>
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<td>Holly &gt; Claire &gt; Twiggy &gt; Jill</td>
<td>Ray &gt; Jake &gt; Elwood &gt; Curtis</td>
</tr>
</tbody>
</table>
Boy proposing

the boys
- Jake
  - Holly > Claire > Twiggy > Jill
- Elwood
  - Claire > Jill > Twiggy > Holly
- Curtis
  - Twiggy > Jill > Holly > Claire
- Ray
  - Holly > Claire > Twiggy > Jill

and girls
- Twiggy
  - Jake > Elwood > Curtis > Ray
- Claire
  - Jake > Curtis > Elwood > Ray
- Jill
  - Ray > Curtis > Elwood > Jake
- Holly
  - Ray > Jake > Elwood > Curtis

(Immorlica/Hartline)
Boy proposing

the boys

Jake
Holly > Claire > Twiggy > Jill

Elwood
Claire > Jill > Twiggy > Holly

Curtis
Twiggy > Jill > Holly > Claire

Ray
Holly > Claire > Twiggy > Jill

and girls

Twiggy
Jake > Elwood > Curtis > Ray

Claire
Jake > Curtis > Elwood > Ray

Jill
Ray > Curtis > Elwood > Jake

Holly
Ray > Jake > Elwood > Curtis

(Immorlica/Hartline)
Boy proposing

Jake
Holly > Claire > Twiggy > Jill

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Claire > Jill > Twiggy > Holly

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Twiggy > Jill > Holly > Claire

Ray
Holly > Claire > Twiggy > Jill

Twiggy
Jake > Elwood > Curtis > Ray

Claire
Jake > Curtis > Elwood > Ray

Jill
Ray > Curtis > Elwood > Jake

Holly
Ray > Jake > Elwood > Curtis

(Immorlica/Hartline)
Girl-Proposing Deferred Acceptance

Round 1  Round 2  Stop! (Stable)

Jake  Holly > Claire > Twiggy > Jill  Twiggy
Elwood  Claire > Jill > Twiggy > Holly  Claire
Curtis  Twiggy > Jill > Holly > Claire  Jill
Ray  Holly > Claire > Twiggy > Jill  Holly

Jake > Elwood > Curtis > Ray
Jake > Curtis > Elwood > Ray
Ray > Curtis > Elwood > Jake
Ray > Jake > Elwood > Curtis

(Immorlica/Hartline)

Not unique!

Jake  Holly > Claire > Twiggy > Jill  Twiggy
Elwood  Claire > Jill > Twiggy > Holly  Claire
Curtis  Twiggy > Jill > Holly > Claire  Jill
Ray  Holly > Claire > Twiggy > Jill  Holly

Jake > Elwood > Curtis > Ray
Jake > Curtis > Elwood > Ray
Ray > Curtis > Elwood > Jake
Ray > Jake > Elwood > Curtis

(Immorlica/Hartline)
Revenge of the LP!

LINEAR PROGRAMMING BRINGS MARITAL BLISS

John H. Vande Vate
School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA

Received March 1988
Revised November 1988

A stable matching is an assignment of $n$ men to $n$ women so that no two people prefer each other to their respective spouses. This paper describes the convex hull of the incidence vectors of stable matchings. With this description, one may solve the optimal stable marriage problem as a linear program.

stable matching • linear programming

1. Introduction

The stable marriage problem asks whether there is a matching of $n$ men to $n$ women so that no two people prefer each other to their respective

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Real-world Matching Markets

National Residency Matching Program (NRMP). School Choice (Boston and New York).

- Adoption of student-proposing NMRP in 1998
- Easier for students
- Practical concern: couples with preferences on pairs of positions
- “Boston mechanism” was not stable or truthful
- Fix: adopt student-proposing DA
Real-world Matching Markets

**National Residency Matching Program (NRMP):**
- Hospitals
- Medical Interns
- Adoption of student-proposing NMRP in 1998
- Easier for students
- Practical concern: couples with preferences on pairs of positions

**School Choice (Boston and New York):**
- Schools
- Students
- “Boston mechanism” was not stable or truthful
- Fix: adopt student-proposing DA
- Easier, more fair, and allow for policy advice
- Practical concern: priorities for schools (siblings, walk zones)

One-sided Matching (strict prefs)

- Each agent “owns” an item (dorm rooms?)
- Strict preferences
- Solutions:
  - Random serial dictatorship (RSD)
  - Top trading cycle algorithm
Top trading cycles

1. Each agent points to her most preferred house that is still on the market.

2. Select a cycle and trade, agents that trade leave the market.

(Hartline/Immorlica)

One-Sided Matching (0/1 prefs)
Kidney disease

- Kidney failure serious medical problem
- Preferred treatment: kidney transplant
  - Cadaver kidneys
  - Donation from live patient
  - Must be blood- and tissue-type compatible

As of early 2016:

121,274 people waiting for a kidney transplant in the US.

http://optn.transplant.hrsa.gov
In 2008,

10,526 patients received cadaver kidneys.

4,857 patients received live donor kidneys.

5,920 patients died or became too sick for a transplant.

(Hartline/Immorlica)

Econ 101

• Trade for $$

I have an extra kidney.

I need a kidney. My value for it is my value for my life.

(Hartline/Immorlica)
Legality

Section 301 of the National Organ Transplant Act, “Prohibition of organ purchases” imposes criminal penalties on any person who

“knowingly acquire[s], receive[s], or otherwise transfer[s] any human organ for valuable consideration for use in human transplantation”

Kidney exchanges
(APD, UNOS, NKR)

1. Pairs register in database.
2. Form a graph, representing possible compatibilities
3. Feasible matchings found
4. Additional medical tests (“cross match”)
5. Transplants performed
Match Offer Process

KX via top-trading cycles?

- No: 0/1 rather than strict preferences
- Limits on cycle lengths
Kidney-Paired Donation

- Find vertex-disjoint cycles of length \( \leq k \) that cover as many vertices as possible

\[
\begin{align*}
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
3 & & 4
\end{array}
\quad & \quad
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
3 & & 4
\end{array}
\quad & \quad
\begin{array}{ccc}
1 & \rightarrow & 2 \\
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3 & & 4
\end{array}
\end{align*}
\]

\( k=2 \) \hspace{1cm} \( k=3 \) \hspace{1cm} \( k=3 \)

Special case: \( k=2 \)

- If edges go in both directions, replace by undirected edge
- Remove other edges

\[
\begin{align*}
\begin{array}{ccc}
1 & \rightarrow & 2 \\
\downarrow & & \downarrow \\
3 & & 4 \\
\downarrow & & \downarrow \\
5 & & 2
\end{array}
\end{align*}
\]

- Find \textit{max cardinality matching} (max #edges, every vertex incident on at most one edge)
- Edmond’s algorithm (poly time.)
Complexity

• \( k = 2 \): in \( \mathcal{P} \) by Edmonds alg
• \( k = \infty \) in \( \mathcal{P} \)
  
  (via a reduction to a maximum weight perfect matching problem)

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**K=\( \infty \): Edge formulation**

\[
\begin{align*}
\max_y & \sum_{(i,j) \in E} y_{ij} \\
\text{s.t.} & \sum_j y_{ij} \leq 1, \quad \forall i \\
& \sum_j y_{ij} = \sum_i y_{ji}, \quad \forall i \\
& y_{ij} \geq 0
\end{align*}
\]

Optimal solutions non-fractional!
k > 2 (not not infinite)

Complexity

• $k = 2$: in $P$ by Edmonds alg
• $k = \infty$ in $P$
• But $k = 3, 4, 5, \ldots$: $NP$-hard!
Cycle formulation (IP)

Doing better

• Cycle size limited because of logistical and ethical concerns

• What can be done?
Altruistic Non-simultaneous Donor Chains

- “Good Samaritan” donors. Enable (long) chains

- New computational challenge. Can reduce “cycles and chains” to “cycles” via zero weight back edges. Allow longer cycles, but ONLY if they involve an altruistic donor.
New clearing problem

- Can only solve 100 pairs with $k \leq 20$
- Active research into this problem
  - Note: cannot use unrestricted length cycles, because may not include an altruistic donor

National Kidney Register

Completed 500+ transplants
Entirely “altruistic non-simultaneous chains”

- 60 person, 30 transplant chain (8/11-12/11)

Extensions

- Dynamics: deciding how to batch the clearing decision (more matches vs fairness)
- Incentive considerations: How to get hospitals to share lists in order to promote better matching?


Other applications

- Peerflix (DVDs)
- Read It Swap It (books)
- Intervac (holiday houses)
- National odd shoe exchange

(Conitzer)