AM 121: Intro to Optimization Models and Methods
Fall 2016

Lecture 16: More cuts, Branch and Cut, other tricks…

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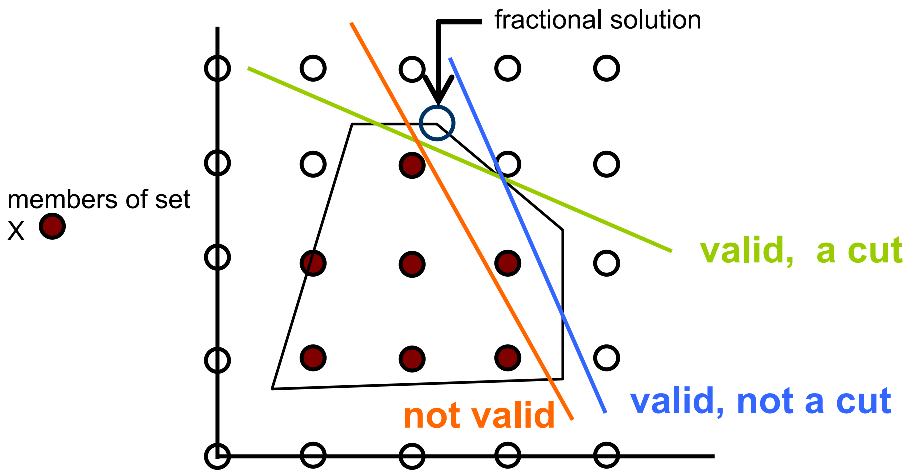
Lesson Plan

- Review Chvátal-Gomory cuts
- Cover cuts
- Branch-and-cut
- Other tricks
  - Preprocessing
  - Primal heuristics
  - SOS branching

Jensen & Bard: 8.5

- **Definition**. An inequality $a^T x \leq b$ is **valid** for set $X \subseteq R^n$ if $a^T x \leq b$ for all $x \in X$.
- **Defn**. A **cut** is a valid inequality that **separates** the current fractional solution x^*.



Chvátal-Gomory inequalities

- Integer, non-negative decision variables
- Consider $X = P \cap Z^2$, where P is given by

$$7x_1 - 2x_2 \leq 14$$
$$x_2 \leq 3$$
$$2x_1 - 2x_2 \leq 3$$
$$x \geq 0$$

- Valid to combine with non-neg weights u=(2/7, 37/63, 0):

$$2x_1 + 1/63 x_2 \leq 121/21$$

- Valid to use non-neg of x to round coefficients on LHS down to nearest integer:

$$2x_1 + 0x_2 \leq 121/21$$

- Valid to use integrality of LHS to round down RHS:

$$2x_1 \leq 5$$

Example (Gomory's algorithm)

- In addition to integer, non-negative decision variables, needs integer coefficients and integer RHs.

- Consider the IP

$$
\begin{aligned}
z = \max\ & 4x_1 - x_2 \\
& 7x_1 - 2x_2 \leq 14 \\
& x_2 \leq 3 \\
& 2x_1 - 2x_2 \leq 3 \\
& x_1, \quad x_2 \geq 0, \text{ integer}
\end{aligned}
$$

> - For row i with fractional RHS, the CG cut is $\sum_{j \in B'} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor)\, x_j \geq \bar{b}_i - \lfloor \bar{b}_i \rfloor$

- $$
\begin{aligned}
z \quad & + 4/7\, x_3 + 1/7\, x_4 && = 59/7 \\
x_1 \quad & + 1/7 x_3 + 2/7\, x_4 && = 20/7 \\
x_2 \quad & + x_4 && = 3 \\
& -2/7\, x_3 + 10/7 x_4 + x_5 && = 23/7
\end{aligned}
$$

- $B=\{1, 2, 5\}$

- For row i with fractional RHS, the CG cut is $\sum_{j\in B'} (\bar{a}_{ij}-\lfloor\bar{a}_{ij}\rfloor)\, x_j \geq \bar{b}_i-\lfloor\bar{b}_i\rfloor$

- $z \quad + \tfrac{4}{7} x_3 + \tfrac{1}{7} x_4 = 59/7$

  $x_1 \quad + 1/7 x_3 + 2/7 x_4 = 20/7$

  $x_2 \quad + x_4 = 3$

  $-2/7\, x_3 + 10/7 x_4 + x_5 = 23/7$

- B={1, 2, 5}
- Add cut $1/7 x_3 + 2/7 x_4 \geq 6/7$
- Add excess variable x_6

- Re-optimize. New optimal tableau:

  $z \quad +\tfrac{1}{2} x_5 + 3x_6 = 15/2$

  $x_1 \quad +x_6 = 2$

  $x_2 \quad -\tfrac{1}{2} x_5 + x_6 = \tfrac{1}{2}$

  $x_3 \quad -x_5 - 5x_6 = 1$

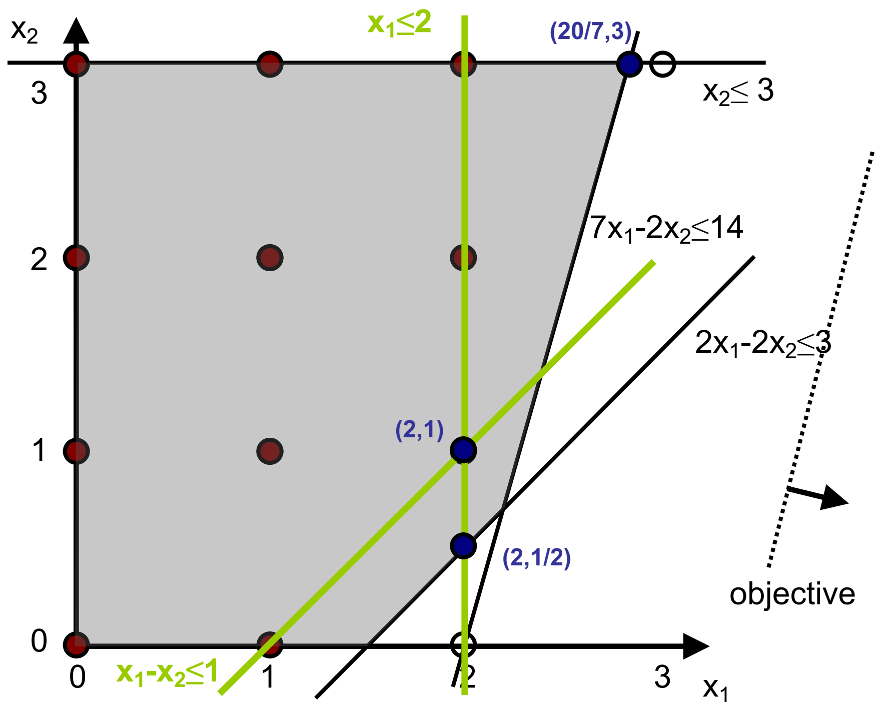
  $x_4 + \tfrac{1}{2} x_5 + 6x_6 = 5/2$

- B={1,2,3,4} and $x^*=(2,1/2,1, 5/2, 0, 0)$
- Add cut $\tfrac{1}{2} x_5 \geq \tfrac{1}{2}$
- Add excess variable x_7

- Re-optimize. New optimal tableau:

$$\begin{array}{lllllllll} z & & & & & & +3x_6 & +\ x_7 & = 7 \\ & x_1 & & & & & +x_6 & & = 2 \\ & & x_2 & & & & +x_6 & -\ x_7 & = 1 \\ & & & x_3 & & & -5x_6 & -2x_7 & = 2 \\ & & & & x_4 & & +6x_6 & +\ x_7 & = 2 \\ & & & & & x_5 & & -\ x_7 & = 1 \end{array}$$

- $x^*=(2, 1, 2, 2, 1, 0, 0)$



Cover inequalities

- Defined for problems in which the feasible space is a “0-1 Knapsack set”. Example:
- $X=\{x\in\{0,1\}^7: 11x_1+6x_2+6x_3+5x_4+5x_5+4x_6+x_7 \le 19\}$

0/1 variables

positive coefficients
positive RHS

- In general, we allow binary IPs with multiple such “knapsack rows” but throughout our discussion we will assume a single such row.

Cover inequalities

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0/1 variables

positive coefficients
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- Example cover inequalities for X are:

$x_1 + x_2 + x_3 \le 2$ Using {0,1} property!

$x_1 + x_2 + x_6 \le 2$

$x_1 + x_5 + x_6 \le 2$

$x_3 + x_4 + x_5 + x_6 \le 3$

Cover inequalities

- For $X = \{x \in \{0,1\}^n : \sum_j a_j x_j \leq b\}$, with $a_j \geq 0$, $b \geq 0$.
- $N=\{1,\ldots,n\}$
- **Defn**. A set $C \subseteq N$ is a **cover** if $\sum_{j \in C} a_j > b$. A cover is **minimal** if $C \setminus \{j\}$ is not a cover for any $j \in C$.
- **Proposition.** If C is a cover, the **cover inequality** $\sum_{j \in C} x_j \leq |C|-1$ is valid.
- **Proof.** Consider $x \in \{0,1\}^n$ with $\sum_{j \in C} x_j = |C|$. Have $\sum_j a_j x_j \geq \sum_{j \in C} a_j x_j = \sum_{j \in C} a_j > b$. Not in X.

Any x that violates cover is infeasible, not in X. So, cover inequality is valid.

Back to the examples

- $X=\{x \in \{0,1\}^7 : 11x_1+6x_2+6x_3+5x_4+5x_5+4x_6+x_7 \leq 19\}$

0/1 variables

positive coefficients
positive RHS

- Example cover inequalities for X are:

  $C=\{1,2,3\}$: $x_1 + x_2 + x_3 \leq |C|-1=2$

  $C=\{1,2,6\}$: $x_1 + x_2 + x_6 \leq |C|-1=2$

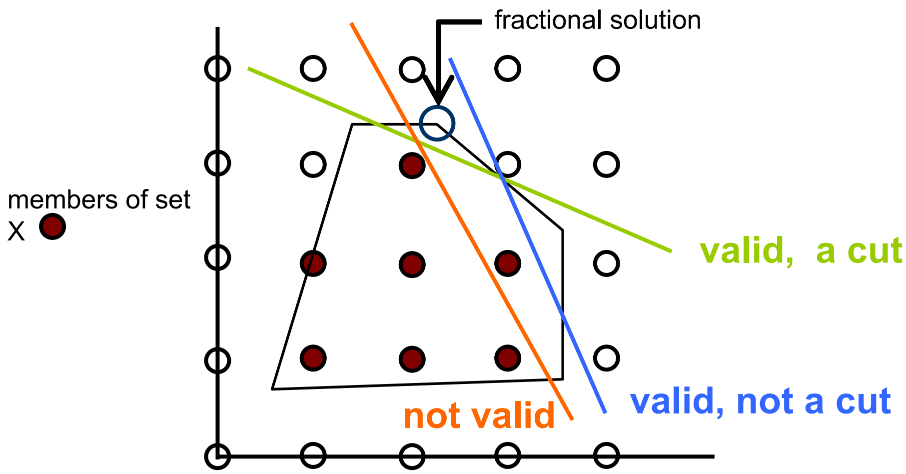
Review: Cover inequalities

- For $X = \{x \in \{0,1\}^n : \sum_j a_j x_j \leq b\}$, with $a_j \geq 0$, $b \geq 0$.
- Given a cover set C, the **cover inequality** is
  $\sum_{j \in C} x_j \leq |C|-1$.
- Equivalently, can write a cover inequality as
  $\sum_{j \in C} (1 - x_j) \geq 1$

- A cover set C has the property that $\sum_{j \in C} a_j > b$.

How can we use Cover inequalities?

- **Definition**. An inequality $a^T x \leq b$ is **valid** for set $X \subseteq R^n$ if $a^T x \leq b$ for all $x \in X$.
- **Defn**. A **cut** is a valid inequality that **separates** the current fractional solution x^*.



Separation with Cover inequalities

- Can write cover inequality as

  $\sum_{j \in C} (1 - x_j) \geq 1$ ("at least one not used")
- Given frac x*, find valid C with $\sum_{j \in C}(1-x^*_j)<1$

- Formulate an IP! Let z_j denote whether $j \in C$.

  $\beta = \min \sum_{j \in N} (1-x^*_j) z_j$

  s.t. $\sum_{j \in N} a_j z_j > b$ (valid cover)

  $z \in \{0,1\}^n$
- If $\beta \geq 1$, then x* satisfies all cover inequalities.
- If $\beta < 1$, then we find a cut.

- Note: this IP is nonstandard (*strict inequality*.) But can replace with $\sum_{j \in N} a_j z_j \geq b+1$ when coeffs are integral.

Example use of a Cover Inequality

- $X=\{x\in\{0,1\}^6: 45x_1+46x_2+79x_3+54x_4+53x_5+125x_6\leq 178\}$, fractional $x^*=(0, 0, 3/4, 1/2, 1, 0)$
- Solve:

  $\beta=\min z_1 + z_2 + \frac{1}{4} z_3 + \frac{1}{2} z_4 + 0z_5 + z_6$

  s.t. $45z_1+46z_2+79z_3+54z_4+53z_5+125z_6>178$

  $z\in\{0,1\}^6$
- $z^*=(0,0,1,1,1,0)$ with $\beta=3/4$
- Conclude that $x_3 + x_4 + x_5 \leq 2$ is a cut for x^*.

- *Why is this useful*? Because we can try to solve the "auxiliary problem" heuristically; give up if too hard.

Two Variations on Cover Inequalities

- Extended Cover Inequalities
- Lifted Cover Inequalities

Extended Cover Inequalities

- $X=\{x\in\{0,1\}^7: 11x_1+6x_2+6x_3+5x_4+5x_5+4x_6+x_7\leq 19\}$
- Cover inequality: $x_3 + x_4 + x_5 + x_6 \leq 3$
- Extended cover ineq: $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$
- Still valid. Why? Any four variables have sum coefficients > 19. (And so $x\in X$ must satisfy)

- **Rule**: Can **extend** *cover set* C to *extended cover set* E(C), where

  $$E(C)= C\cup\{ j : a_j \geq a_i \text{ for all } i\in C\}.$$
- Extended cover inequality: $\sum_{j\in E(C)} x_j \leq |C|-1$ is valid, and stronger than cover inequality.

Two Variations on Cover Inequalities

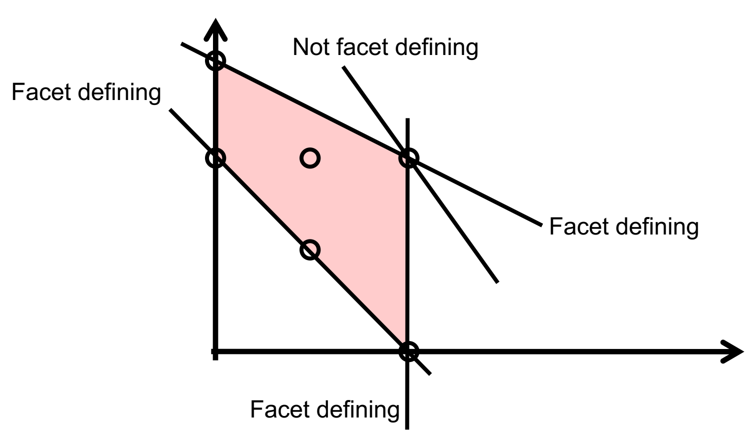
- Extended Cover Inequalities
- Lifted Cover Inequalities

Lifted Cover Inequalities

- $X=\{x\in\{0,1\}^7: 11x_1+6x_2+6x_3+5x_4+5x_5+4x_6+x_7\leq 19\}$
- **Lifting**: Given $x_2+x_3+x_4+x_5+x_6\leq 3$, find a valid inequality $\boldsymbol{\alpha_1 x_1}+x_2+x_3+x_4+x_5+x_6\leq 3$, for some $\alpha_1>0$.
- Valid for any soln with $x_1=0$.
- What about solns with $x_1=1$? Well, any $x\in X$ with $x_1=1$ satisfies: $6x_2+6x_3+5x_4+5x_5+4x_6+x_7\leq 8$ (*)

- **Idea**: pick maximum α_1 s.t. $\alpha_1+x_2+x_3+x_4+x_5+x_6\leq 3$ for all **integer** solutions satisfying (*)
- Set $\alpha_1= 3 - \max_x \{x_2+x_3+x_4+x_5+x_6:$
  $6x_2+6x_3+5x_4+5x_5+4x_6+x_7\leq 8,$
  $x\in\{0,1\}^6 \} = 3 - 1 = 2$
- **Lifted cover inequal**: $2x_1+x_2+x_3+x_4+x_5+x_6 \leq 3$

- **Theorem**. Lifted cover inequalities are "facet-defining" for *conv*(X) when X is {0,1}-knapsack set.
- (*Informal*) An inequality is **facet defining** if it defines the face of a polyhedron. Consider:



Lifted Cover Inequalities

- Complete for IPs with constraints that are 0-1 knapsack sets.
- But, rather than use them to construct the convex hull we will do something else…

- Let's see this next. How to use lifted cover inequalities and other approaches?

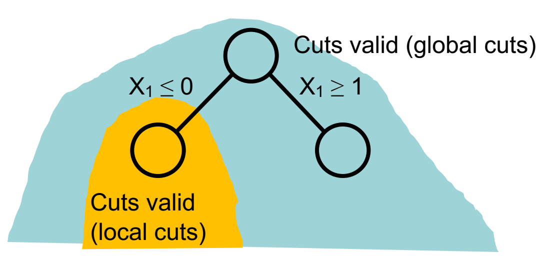
- We've seen two families of cuts:
  - Chvátal-Gomory cuts for IPs with integer coefficients and integer RHS values
  - (Lifted, extended) cover cuts for IPs with {0,1} vars and strictly positive coeffs and strictly positive RHS values
- There are many others:
  - Flow cover
  - Mixed integer rounding
  - Odd hole
  - …
- *How can we use these?*

The Branch-and-Cut method

- Generalizes Branch-and-Bound.
- If cannot prune a node, can try to find a **cut.**
- Strengthen the formulation; re-solve the LP, may now be able to prune by bound.
- If no cuts are found, then branch.

The Branch-and-Cut method

- Branch-and-Bound + **cuts generated throughout the tree**.
- Cuts generated at root are "global," cuts at subproblems "local." Place into a "cut pool."
- **Goal**: reduce the number of search nodes by improving bounds



Example Application

- Winner determination problem in an auction, e.g. selling lots of wine.
- 10 bidders, 5 different wines (≈100 cases each).
- Bidders wants either wine 1, or wine 2, or wine 3…
- Goal: maximize total revenue

Bidders		**Goods**				
		A	*B*	*C*	*D*	*E*
1	demand	91	1	21	…	
	value	110	16	25	…	
2	demand	54	53	…		
	value	65	69	…		
3	demand	3	87	…		
	value	19	93	…		
Supply		91	87	109	88	64

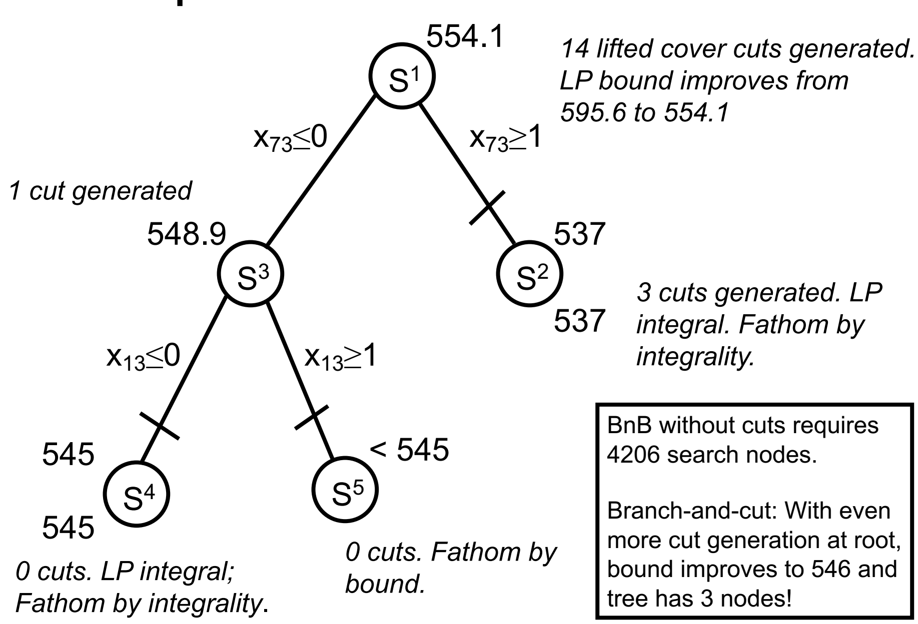
Winner Determination Problem

- Bidders i, goods j with supply b_j
- c_{ij} value to i for a_{ij} units of good j
- x_{ij}=1 if select bid from i on j as a winner

$$\max \textstyle\sum_i \sum_j c_{ij} x_{ij} \qquad \text{(Generalized Assignment)}$$
$$\text{s.t. } \textstyle\sum_j x_{ij} \le 1 \;\; \text{for all } i = 1,\ldots,m$$
$$\textstyle\sum_i a_{ij}x_{ij} \le b_j \;\; \text{for all } j = 1,\ldots,n$$
$$x_{ij} \in \{0,1\}$$

Has "0/1 knapsack set" structure. Can use Cover cuts.

Example Search Tree



BnB without cuts requires 4206 search nodes.

Branch-and-cut: With even more cut generation at root, bound improves to 546 and tree has 3 nodes!

Historical Context

- Use of cutting planes as a practical tool for solving IPs was *abandoned* in 1960s and ‘70s!!!
- Revisited in early ‘70s following **NP**-completeness theory (interest in problem structure and facets.)
- Balas, Wolsey, Johnson, Padberg developed **cover inequalities** and “lifting” in mid ‘70s.
- **Branch-and-cut** developed in early ‘80s and introduced into commercial solvers in ‘00s.
- Provides **order of magnitude** improvement in solve times.

Other important components

- Preprocessing
- Primal heuristics
- SOS branching

Preprocessing

- Detect and eliminate redundant constraints and variables, tighten bounds where possible

Preproc.: Isolating and Fixing

- Consider LP: (Wolsey pp.104)
- max $2x_1 + x_2 - x_3$

  $5x_1 - 2x_2 + 8x_3 \leq 15$ (a)

  $8x_1 + 3x_2 - x_3 \geq 9$ (b)

  $x_1 + x_2 + x_3 \leq 6$ (c)

  $0 \leq x_1 \leq 3$; $0 \leq x_2 \leq 1$; $1 \leq x_3$
- Isolate x_1 in (a) to obtain $5x_1 \leq 15+2x_2 - 8x_3 \leq 15+2-8=9$, and so $x_1 \leq 9/5$.
- Isolate x_3 in (a) to obtain $x_3 \leq 17/8$. Not useful to isolate x_2.
- Can also isolate x_1 in (b); obtain $x_1 \geq 7/8$.
- **Repeat**: improve bound on x_3 from (a) by using $x_1 \geq 7/8$; obtain $x_3 \leq 101/64$
- Find constraint (c) is redundant, using $x_1 \leq 9/5$ and $x_3 \leq 101/64$, together with $x_2 \leq 1$.

“Fixing” values of variables

The modified LP is:

- max $2x_1 + x_2 - x_3$

  s.t. $5x_1 - 2x_2 + 8x_3 \leq 15$

  $8x_1 + 3x_2 - x_3 \geq 9$

  $7/8 \leq x_1 \leq 9/5$; $0 \leq x_2 \leq 1$; $1 \leq x_3 \leq 101/64$

- **Fixing:** In optimal solution, have $x_2 = 1$; and $x_3 = 1$.
- Why? x_2 has positive coeff in obj, and increasing makes constraints less tight. Similarly, best to decrease x_3.

- Obtain: $\max\{2x_1 : 7/8 \leq x_1 \leq 9/5\}$. Easy to solve!

Other important components

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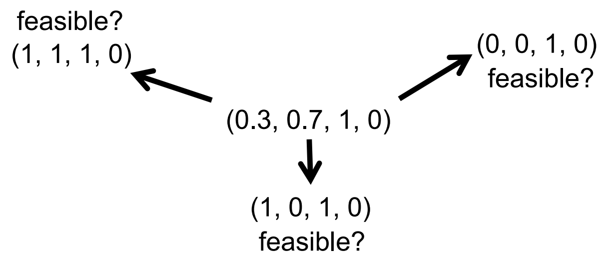
- SOS branching

Primal heuristics

- At a fractional search node, use a heuristic approach to try to find an integral solution
- For example, round the solution or do a directed "neighborhood search."



- May be able to improve incumbent.

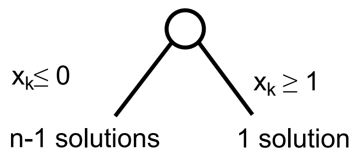
Other important components

- Preprocessing
- Primal heuristics
- SOS branching

SOS branching

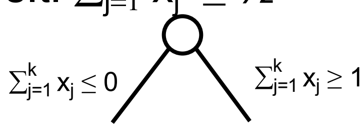
- Consider constraints $\sum_{j=1}^{n} x_j=1$, $x_j \in \{0,1\}$
- Typical to branch on fractional x_k^* :

unbalanced



- "Special-ordered sets" branching. Pick smallest k s.t. $\sum_{j=1}^{k} x_j^* \geq ½$

balanced



Example: $x^*=(0,0.2,0.4,0,0.4)$. Branch $x_1 + x_2 + x_3 \leq 0$, $x_1 + x_2 + x_3 \geq 1$

Summary: Advanced IP solving

- MIPs are a lot harder to solve than LPs!
- But IP technology is sophisticated:
  - Dual simplex pivots
  - Tight formulations
  - Automated cut generation
  - Primal heuristics
  - Node and variable selection heuristics
- Problems with 100,000s variables and 10,000s constraints can be solved in commercially viable times.