AM 121: Intro to Optimization Models and Methods
Fall 2016

Lecture 15: Cutting plane methods

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Lesson Plan

- Cut generation and the separation problem
- Cutting plane methods
- Chvátal-Gomory cuts
- Gomory's cutting plane algorithm

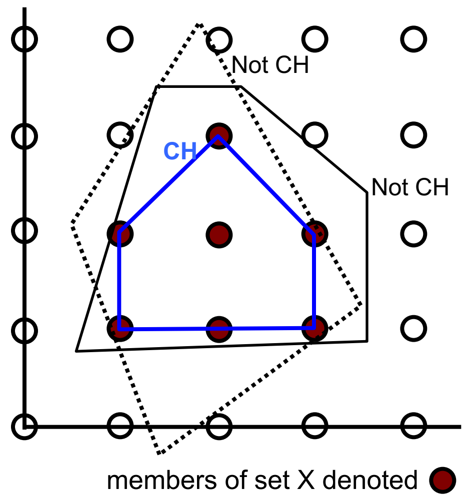
Textbook Reading: 8.4

Convex Hull

- **Definition.** Given set $X \subseteq Z^n$, the **convex hull** of $X=\{x^1,\ldots,x^t\}$ is *conv*$(X)=\{x: x=\sum_{k=1}^{t} \lambda_k x^k, \sum_{k=1}^{t}\lambda_k=1, \lambda_k \geq 0$ for all k$\}$

- **Prop.** *conv*(X) is a polyhedron.
- **Prop.** Extreme points of *conv*(X) all lie in X.
- **Prop.** Can solve IP via solving LP on *conv*(X).

Challenge: May need expon. number of inequalities to describe *conv*(X).



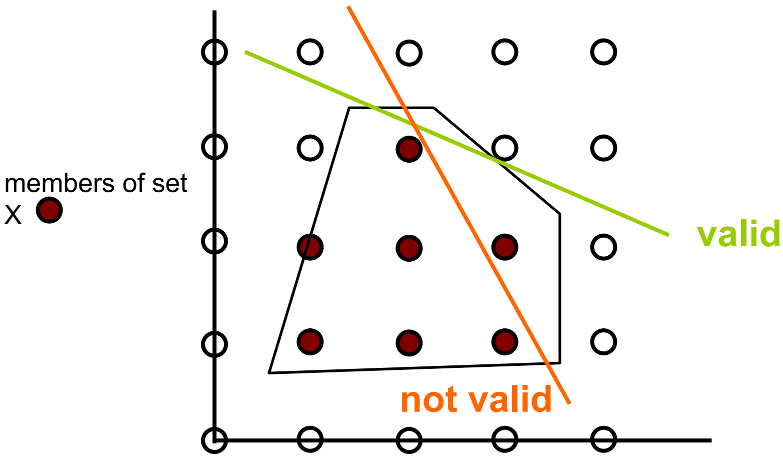
Cut Generation

- Q: What else can we do to improve the strength of formulations?
- A: *Automatically generate new inequalities ("cuts") that approximate the convex hull.*

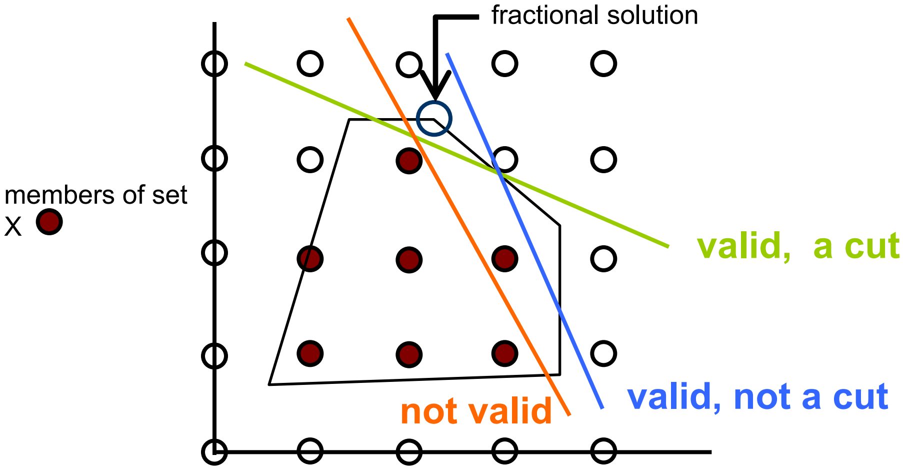
- Why this might be useful:
  – improve branch-and-bound (stronger formulation, and thus improved bounds)
  – provides a completely new way to solve IPs

- **Definition**. An inequality $a^T x \leq b$ is a **valid inequality** for set $X \subseteq R^n$ if $a^T x \leq b$ for all $x \in X$.

- "keeps all integer solutions"

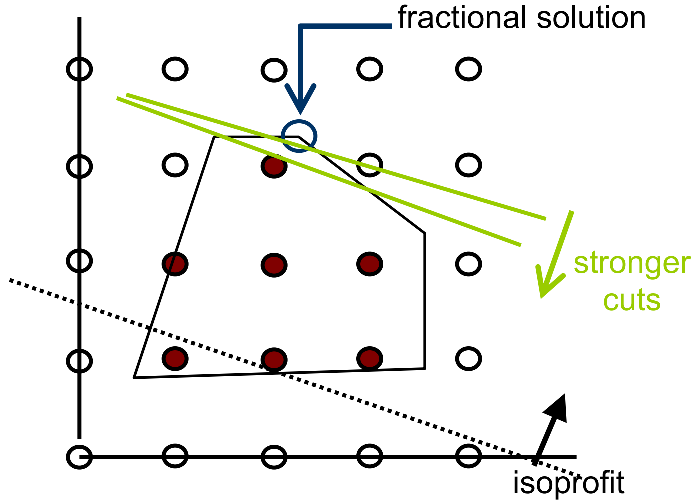


- **Definition**. An inequality $a^T x \leq b$ is a **valid inequality** for set $X \subseteq R^n$ if $a^T x \leq b$ for all $x \in X$.
- **Defn**. A **cut** is a valid inequality that **separates** the current fractional solution x^*.



Cut strength

- **Definition**. Cut *c* is stronger than cut *c*' if $z^{LP} < z^{LP'}$, where LP includes c and LP' includes c'.



The Cutting-Plane Method

- Step 1: Solve LPR. Get x^*.
- Step 2: If x^* integral, **stop**. Else, find a valid inequality that excludes x^* (a "cut")
- Step 3: Go to Step 1.

➔ keep strengthening the formulation until the IP is solved

Questions

- How to generate **strong cuts**, and quickly?
- Will a cutting-plane algorithm always **terminate** with the **optimal IP solution**?

- For warm-up, let's look at examples of valid inequalities.

Example 1

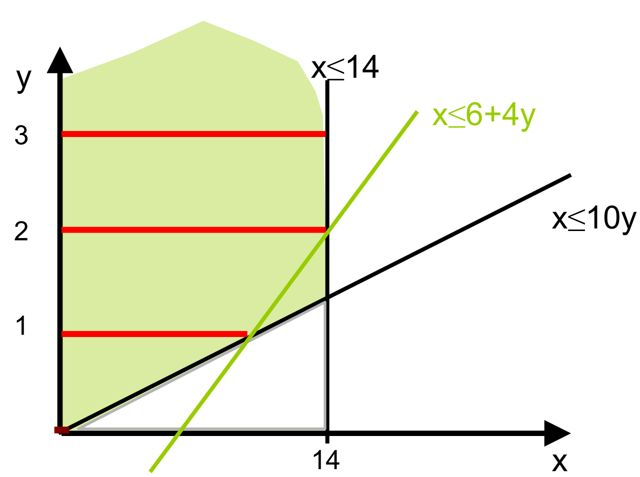
- $X = \{x \in \{0,1\}^5 : 3x_1 – 4x_2 + 2x_3 – 3x_4 + x_5 \leq -2\}$
- First: If $x_2=x_4=0$, the LHS ≥ 0, and the solution is infeasible.
  - **By integrality**, a valid inequality is $x_2+x_4 \geq 1$.
  - Does not remove any $x \in X$, so valid. Also: removes fractional solns eg., $x=(0,1/3, 0, 1/3, 0)$.
- Second: If $x_2=0$ but $x_1=1$, the LHS ≥ 0, and the solution is infeasible.
  - **By integrality**, a valid inequality is $x_1 \leq x_2$.
  - Does not remove any $x \in X$, so valid. Also: removes fractional solns e.g., $x=(2/6,1/6, 0, 1, 0)$.

Example 2

- X = {(x,y) : $x \le 9999y$, $0 \le x \le 5$, $x \in Z$, $y \in \{0,1\}$ }
- X = {(0,0),(0,1),(1,1),(2,1),(3,1),(4,1),(5,1)}
- A valid inequality is $x \le 5y$.
  - Does not remove any solutions in X, so valid.
  - Also: removes fractional solutions such as (1,0.1).

Example 3

- X = {(x,y) : $x \le 10y$, $0 \le x \le 14$, $y \in Z_{\ge 0}$}
- $x \le 6+4y$ is valid. Doesn't remove any solns in



Example 4 (Chvátal-Gomory inequality)

- Consider $X = P \cap Z^2$, where P is given by

  $7x_1 - 2x_2 \leq 14$

  $x_2 \leq 3$

  $2x_1 - 2x_2 \leq 3$

  $x \geq 0$

Notes:
(a) Important that X is integers
(b) Applies to "$\leq$" inequalities
(c) The result of steps (1) and (2) is implied by P and is weaker. But the integrality on the LHS allows step 3 to tighten!

- Valid to form a non-neg, linear combination of inequalities. Eg., multipliers u=(2/7, 37/63, 0). Obtain:

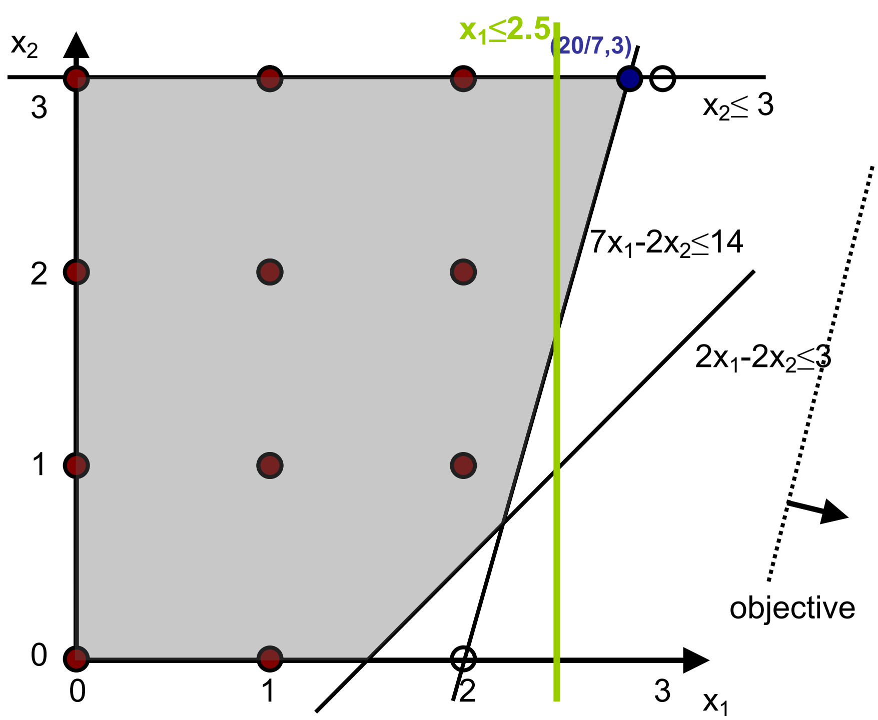
$$2x_1 + 1/63x_2 \leq 121/21$$

- Since $x \geq 0$, valid to round coeffs on LHS down:

$$2x_1 + 0x_2 \leq 121/21$$

- Because LHS is **integral** for all $x \in X$, valid to round RHS down to nearest integer. Obtain:

$$2x_1 + 0x_2 \leq 5$$



General Approach to CG Inequality

$X = P \cap Z^n$, $P=\{x \in R^n_{\geq 0}: Ax \leq b\}$, some $u \in R^m_{\geq 0}$

Three steps:

(i) Combine rows of A with **nonnegative weights** u. This is valid for X. Obtain an "$\leq$" inequality.

(ii) Take the **floor of coefficients on LHS**. This is a valid inequality for X because $x \geq 0$.

(iii) LHS has integer value => valid to take the **floor of the RHS.** [The LHS has integer values because coeffs and x are integer.]

- **Theorem**. Given any fractional, extreme point x* of P, there exists multipliers $u \geq 0$ s.t. the CG inequality is a cut (for x*).

=> **CG cuts are complete for IP**!! In principle we can solve IPs by repeated CG cuts.

But:

- How many CG cuts do we need to generate?
- How do we generate the cuts?

Answer: Use Gomory's algorithm

- CG inequalities are valid for any integer program with "$\leq$" and non-neg, integer decision variables.
- Gomory's algorithm uses CG inequalities in a particular way. It can only be applied to an integer program with **integer coeffs** and **integer RHS's.**

(WLOG for rational problem: achieve by rescaling)

- It generates CG cuts, and provably converges to the optimal IP solution.
- Uses the simplex tableau, generates a cut from any row with a fractional RHS value.
- No fractional rows left => a {0,1} solution!

Gomory's cutting plane algorithm

- "Outline of an algorithm for Integer solutions to linear programs," R.E.Gomory, *Bulletin of the American Mathematical Society* **64**, 275-278 (1958)
- "An algorithm for Integer solutions to Linear programs," R.E.Gomory, in R. Graves and P. Wolfe, eds., *Recent advances in Mathematical Programming* McGraw-Hill, 269-302 (1964)
- "Edmonds polytopes and a hierarchy of combinatorial problems," V. Chvatal, *Discr. Math.* **4**, 305–337 (1973)

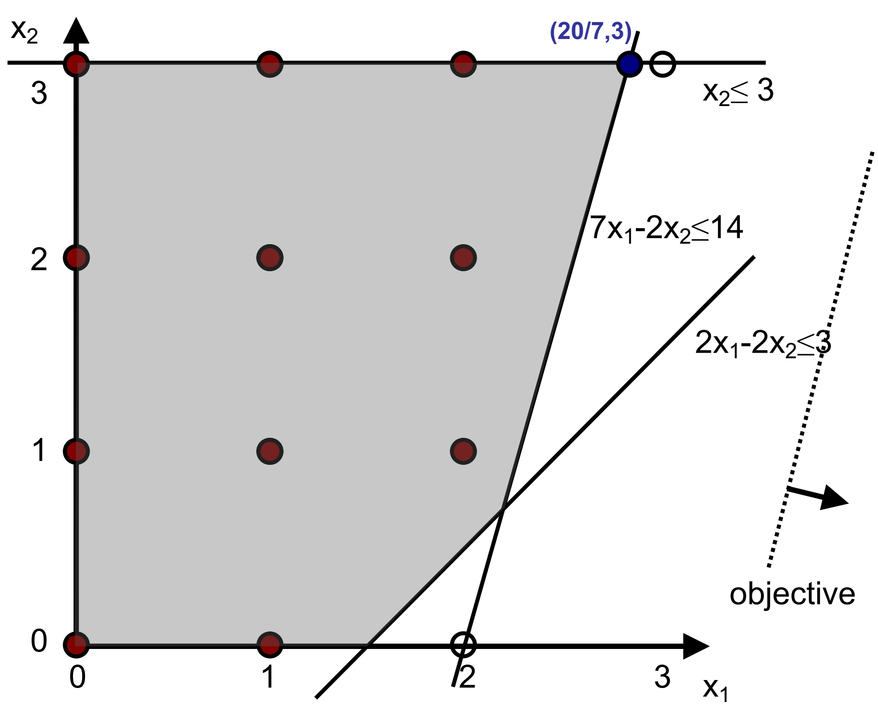
Example (Gomory's algorithm)

- For an IP with $A \in Z^{mn}$ and $b \in Z^m$.
- For example:

(Need integer coefficients, RHS. WLOG for rational problem: achieve by rescaling.)

$$
\begin{aligned}
z = \max\ & 4x_1 - x_2 \\
\text{s.t.}\quad & 7x_1 - 2x_2 \le 14 \\
& x_2 \le 3 \\
& 2x_1 - 2x_2 \le 3 \\
& x_1,\quad x_2 \ge 0, \text{ integer}
\end{aligned}
$$

- Introduce slack variables x_3, x_4 and x_5.
  - A and b integer: therefore, we can insist that slack variables are non-negative, integer.



Generating a CG cut

Optimal tableau:

- $z \quad +4/7\, x_3 + 1/7\, x_4 = 59/7$

  $x_1 \quad +1/7\, x_3 + 2/7\, x_4 = 20/7$

  $x_2 \quad + x_4 = 3$

  $-2/7\, x_3 + 10/7 x_4 + x_5 = 23/7$

- B={1, 2, 5} and x^*=(20/7, 3, 0, 0, 23/7)
- Form a CG-cut from **any row** with a fractional RHS.
  - "u" vector puts 1 on this row, 0 on other rows.
  - apply CG procedure to the "$\leq$" inequality implied by the equality (CG is defined for "$\leq$" inequalities).
- Suppose we choose x_1 row. The CG cut is:

  $x_1 + 0x_3 + 0x_4 \leq 2$

Optimal tableau:

- $z \quad +4/7\, x_3 + 1/7\, x_4 = 59/7$

  $x_1 \quad +1/7\, x_3 + 2/7\, x_4 = 20/7$ (1)

  $x_2 \quad + x_4 = 3$

  $-2/7\, x_3 + 10/7 x_4 + x_5 = 23/7$

- Add CG cut from x_1 row (convenient to negate):

  $-x_1 \geq -2$ (a)

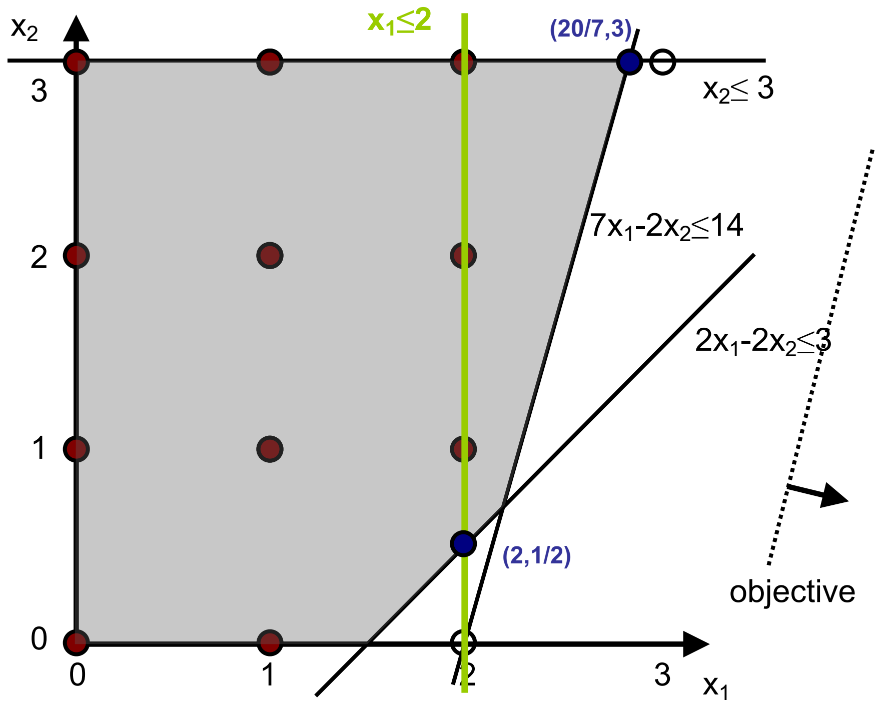
- Will want to isolate current basis. (1) + (a):

  $1/7 x_3 + 2/7 x_4 \geq 6/7$ (b)

- Add non-neg, integer *excess var* $x_6 \geq 0$:

  $1/7 x_3 + 2/7 x_4 - x_6 = 6/7$ (c)

- Note: x_6 integer because LHS-RHS in (a) is integer, also true for (b) since (1) is an equality.



- Add (c) and re-optimize.
- New optimal tableau:

$$
\begin{aligned}
z \quad & + \tfrac{1}{2}x_5 + 3x_6 = 15/2 \\
x_1 \quad & + x_6 = 2 \\
x_2 \quad & - \tfrac{1}{2}x_5 + x_6 = \tfrac{1}{2} \\
x_3 \quad & - x_5 - 5x_6 = 1 \\
x_4 & + \tfrac{1}{2}x_5 + 6x_6 = 5/2
\end{aligned}
$$

- $B=\{1,2,3,4\}$ and $x^*=(2, 1/2, 1, 5/2, 0, 0)$

- Add another CG cut. Use a general rule to go from fractional tableau to a new cut.

General Rule for Deriving a Cut

- For row i with fractional RHS, the CG cut is $\sum_{j\in B'} (\bar{a}_{ij}-\lfloor\bar{a}_{ij}\rfloor)\, x_j \geq \bar{b}_i-\lfloor\bar{b}_i\rfloor$

Example 1:

- $z \quad + 4/7\, x_3 + 1/7\, x_4 = 59/7$

  $x_1 + 1/7 x_3 + 2/7\, x_4 = 20/7$

  $x_2 + x_4 = 3$

  $-2/7\, x_3 + 10/7 x_4 + x_5 = 23/7$

- B={1, 2, 5}
- Cut $1/7 x_3 + 2/7 x_4 \geq 6/7$

General Rule for Deriving a Cut

- For row i with fractional RHS, the CG cut is $\sum_{j\in B'} (\bar{a}_{ij}-\lfloor\bar{a}_{ij}\rfloor)\, x_j \geq \bar{b}_i-\lfloor\bar{b}_i\rfloor$

Example 2:

$z \quad +\tfrac{1}{2}x_5 + 3x_6 = 15/2$

$x_1 \quad +x_6 = 2$

$x_2 \quad -\tfrac{1}{2}x_5 + x_6 = \tfrac{1}{2}$

$x_3 \quad -x_5 - 5x_6 = 1$

$x_4 + \tfrac{1}{2}x_5 + 6x_6 = 5/2$

- B={1, 2, 3, 4}
- Cut $\tfrac{1}{2}x_5 \geq \tfrac{1}{2}$ $(\tfrac{1}{2} = -\tfrac{1}{2}-(\lfloor -\tfrac{1}{2}\rfloor) = -\tfrac{1}{2} - (-1) = \tfrac{1}{2})$

- Add (c) and re-optimize.
- New optimal tableau:

$$
\begin{array}{lllll}
z & & & & +\tfrac{1}{2}x_5 + 3x_6 = 15/2 \\
 & x_1 & & & +x_6 = 2 \\
 & & x_2 & & -\tfrac{1}{2}x_5 + x_6 = \tfrac{1}{2} \\
 & & & x_3 & -x_5 - 5x_6 = 1 \\
 & & & & x_4 + \tfrac{1}{2}x_5 + 6x_6 = 5/2
\end{array}
$$

- $B=\{1,2,3,4\}$ and $x^*=(2, 1/2, 1, 5/2, 0, 0)$
- Add CG cut $\tfrac{1}{2}x_5 \geq \tfrac{1}{2}$
- Bring in integer, non-neg excess var x_7.

$$\tfrac{1}{2}x_5 - x_7 = \tfrac{1}{2} \qquad \text{(d)}$$

- Add (d) and re-optimize. New tableau:

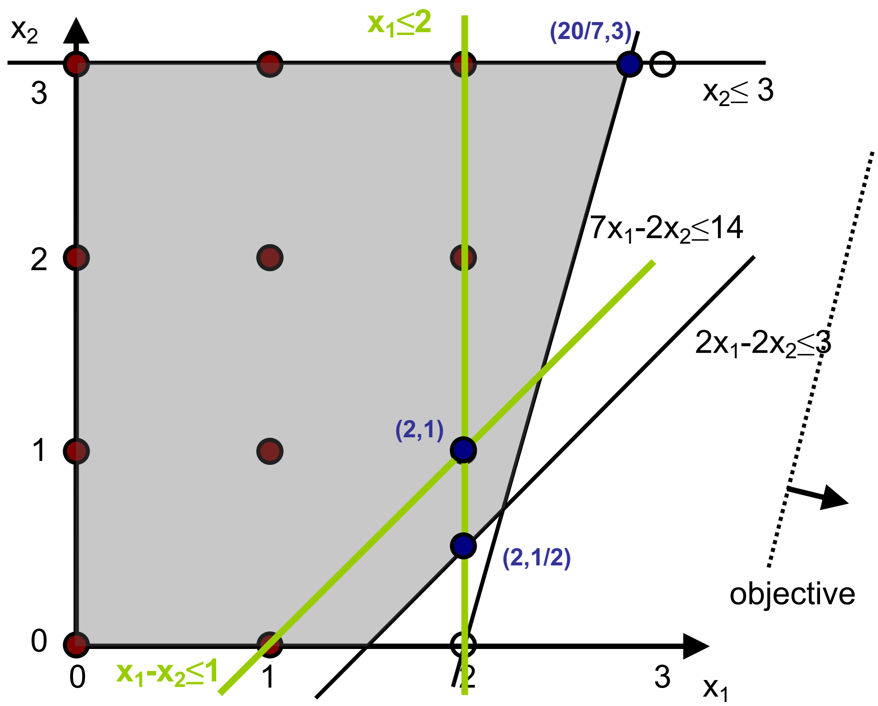
$$
\begin{array}{llllll}
z & & & & & +3x_6 + x_7 = 7 \\
 & x_1 & & & & +x_6 = 2 \\
 & & x_2 & & & +x_6 - x_7 = 1 \\
 & & & x_3 & & -5x_6 - 2x_7 = 2 \\
 & & & & x_4 & +6x_6 + x_7 = 2 \\
 & & & & & x_5 - x_7 = 1
\end{array}
$$

- $x^*=(2, 1, 2, 2, 1, 0, 0)$

- Integral! $(x_1, x_2)=(2,1)$ solves the original IP.

What happened? Visualizing.

- Map the cuts back to the (x_1,x_2) space
- First cut: $x_1 \leq 2$
- Second cut: $\frac{1}{2}x_5 \geq \frac{1}{2}$. Substituting for $x_5=3-2x_1+2x_2$, rearrange and obtain $x_1-x_2 \leq 1$.



Summary: Gomory's algorithm

- **Repeat**:
  - Solve LP.
  - If integral, STOP. Else, generate a CG-cut from **any** row with fractional RHS.

- **Thm**. Gomory's algorithm will solve an IP with *integer coefficients and integer RHS values*.

Gomory's algorithm

- Not of practical interest, but transformative method!
- Can solve IPs without branch and bound, and inspired effort into identifying families of strong cuts.
- With some work, can extend to solve mixed IPs

Summary: Cutting plane method

- Cuts are valid inequalities that separate the current optimal, fractional solution
- The cutting-plane method
- Chvatal-Gomory cuts for IPs (they are complete!)
- Gomory's cutting plane method
  - Solves IPs without branch and bound
  - Works on IPs with integer coefficients and integer RHS.