Lecture 15: Cutting plane methods

Lesson Plan

• Cut generation and the separation problem
• Cutting plane methods
• Chvátal-Gomory cuts
• Gomory’s cutting plane algorithm

Textbook Reading: 8.4
Convex Hull

- **Definition.** Given set $X \subseteq \mathbb{Z}^n$, the **convex hull** of $X = \{x^1, \ldots, x^t\}$ is $\text{conv}(X) = \{x : x = \sum_{k=1}^{t} \lambda_k x^k, \sum_{k=1}^{t} \lambda_k = 1, \lambda_k \geq 0 \text{ for all } k\}$

- **Prop.** $\text{conv}(X)$ is a polyhedron.

- **Prop.** Extreme points of $\text{conv}(X)$ all lie in $X$.

- **Prop.** Can solve IP via solving LP on $\text{conv}(X)$.

**Challenge:** May need exponential number of inequalities to describe $\text{conv}(X)$.

Cut Generation

- **Q:** What else can we do to improve the strength of formulations?

- **A:** *Automatically generate new inequalities (“cuts”) that approximate the convex hull.*

- Why this might be useful:
  - improve branch-and-bound (stronger formulation, and thus improved bounds)
  - provides a completely new way to solve IPs
• **Definition.** An inequality $a^T x \leq b$ is a **valid inequality** for set $X \subseteq \mathbb{R}^n$ if $a^T x \leq b$ for all $x \in X$.

• “keeps all integer solutions”

• **Definition.** An inequality $a^T x \leq b$ is a **valid inequality** for set $X \subseteq \mathbb{R}^n$ if $a^T x \leq b$ for all $x \in X$.

• **Defn.** A **cut** is a valid inequality that **separates** the current fractional solution $x^*$. 

---

![Diagram showing valid and not valid inequalities for a set X.](image-url)
Cut strength

- **Definition.** Cut $c$ is stronger than cut $c'$ if $z^{LP} < z^{LP'}$, where LP includes $c$ and LP' includes $c'$.

The Cutting-Plane Method

- **Step 1:** Solve LPR. Get $x^*$.
- **Step 2:** If $x^*$ integral, **stop**. Else, find a valid inequality that excludes $x^*$ (a “cut”)
- **Step 3:** Go to Step 1.

⇒ keep strengthening the formulation until the IP is solved
Questions

• How to generate strong cuts, and quickly?
• Will a cutting-plane algorithm always terminate with the optimal IP solution?

• For warm-up, let’s look at examples of valid inequalities.

Example 1

• $X = \{x \in \{0,1\}^5 : 3x_1 - 4x_2 + 2x_3 - 3x_4 + x_5 \leq -2\}$
• First: If $x_2 = x_4 = 0$, the LHS $\geq 0$, and the solution is infeasible.
  – By integrality, a valid inequality is $x_2 + x_4 \geq 1$.
  – Does not remove any $x \in X$, so valid. Also: removes fractional solns eg., $x = (0, 1/3, 0, 1/3, 0)$.
• Second: If $x_2 = 0$ but $x_1 = 1$, the LHS $\geq 0$, and the solution is infeasible.
  – By integrality, a valid inequality is $x_1 \leq x_2$.
  – Does not remove any $x \in X$, so valid. Also: removes fractional solns eg., $x = (2/6, 1/6, 0, 1, 0)$.
Example 2

- $X = \{(x,y) : x \leq 9999y, 0 \leq x \leq 5, x \in \mathbb{Z}, y \in \{0,1\}\}$
- $X = \{(0,0),(0,1),(1,1),(2,1),(3,1),(4,1),(5,1)\}$
- A **valid inequality** is $x \leq 5y$.
  - Does not remove any solutions in $X$, so valid.
  - Also: removes fractional solutions such as $(1,0.1)$.

Example 3

- $X = \{(x,y) : x \leq 10y, 0 \leq x \leq 14, y \in \mathbb{Z}_{\geq 0}\}$
- $x \leq 6+4y$ is valid. Doesn’t remove any solns in
Example 4 (Chvátal-Gomory inequality)

- Consider $X = \mathbb{P} \cap \mathbb{Z}^2$, where $\mathbb{P}$ is given by
  
  $\begin{align*}
  7x_1 - 2x_2 &\leq 14 \\
  x_2 &\leq 3 \\
  2x_1 - 2x_2 &\leq 3 \\
  x &\geq 0
  \end{align*}$

- Valid to form a non-neg, linear combination of inequalities. Eg., multipliers $u=(2/7, 37/63, 0)$. Obtain:
  
  $2x_1 + \frac{1}{63}x_2 \leq \frac{121}{21}$

- Since $x \geq 0$, valid to round coeffs on LHS down:
  
  $2x_1 + 0x_2 \leq \frac{121}{21}$

- Because LHS is integral for all $x \in X$, valid to round RHS down to nearest integer. Obtain:
  
  $2x_1 + 0x_2 \leq 5$

Notes:

(a) Important that $X$ is integers
(b) Applies to "$\leq$" inequalities
(c) The result of steps (1) and (2) is implied by $\mathbb{P}$ and is weaker. But the integrality on the LHS allows step 3 to tighten.
General Approach to CG Inequality

\[ X = P \cap \mathbb{Z}^n, P = \{ x \in \mathbb{R}^n_{\geq 0} : Ax \leq b \}, \text{ some } u \in \mathbb{R}^m_{\geq 0} \]

Three steps:
(i) Combine rows of \( A \) with nonnegative weights \( u \). This is valid for \( X \). Obtain an “\( \leq \)” inequality.
(ii) Take the floor of coefficients on LHS. This is a valid inequality for \( X \) because \( x \geq 0 \).
(iii) LHS has integer value => valid to take the floor of the RHS. [The LHS has integer values because coeffs and \( x \) are integer.]

• **Theorem.** Given any fractional, extreme point \( x^* \) of \( P \), there exists multipliers \( u \geq 0 \) s.t. the CG inequality is a cut (for \( x^* \)).

=> CG cuts are complete for IP!! In principle we can solve IPs by repeated CG cuts.

But:
• How many CG cuts do we need to generate?
• How do we generate the cuts?
Answer: Use Gomory’s algorithm

- CG inequalities are valid for any integer program with “≤” and non-neg, integer decision variables.
- Gomory’s algorithm uses CG inequalities in a particular way. It can only be applied to an integer program with integer coeffs and integer RHS’s.
  
  (WLOG for rational problem: achieve by rescaling)

- It generates CG cuts, and provably converges to the optimal IP solution.
- Uses the simplex tableau, generates a cut from any row with a fractional RHS value.
- No fractional rows left => a {0,1} solution!

Gomory's cutting plane algorithm

Example (Gomory’s algorithm)

• For an IP with $A \in \mathbb{Z}^{mn}$ and $b \in \mathbb{Z}^m$.

• For example:

$$z = \max 4x_1 - x_2$$
\[\text{s.t.} \quad 7x_1 - 2x_2 \leq 14\]
\[x_2 \leq 3\]
\[2x_1 - 2x_2 \leq 3\]
\[x_1, \quad x_2 \geq 0, \text{ integer}\]

• Introduce slack variables $x_3$, $x_4$ and $x_5$.
  - $A$ and $b$ integer: therefore, we can insist that slack variables are non-negative, integer.
Generating a CG cut

Optimal tableau:

- $z \quad +\frac{4}{7} x_3 + \frac{1}{7} x_4 = \frac{59}{7}$
- $x_1 \quad +\frac{1}{7} x_3 + \frac{2}{7} x_4 = \frac{20}{7}$
- $x_2 \quad + x_4 = 3$
- $-\frac{2}{7} x_3 + \frac{10}{7} x_4 + x_5 = \frac{23}{7}$

- $B=\{1, 2, 5\}$ and $x^*=(\frac{20}{7}, 3, 0, 0, \frac{23}{7})$
- Form a CG-cut from any row with a fractional RHS.
  - “u” vector puts 1 on this row, 0 on other rows.
  - apply CG procedure to the “≤” inequality implied by the equality (CG is defined for “≤” inequalities).
- Suppose we choose $x_1$ row. The CG cut is:
  - $x_1 + 0x_3 + 0x_4 \leq 2$

Optimal tableau:

- $z \quad +\frac{4}{7} x_3 + \frac{1}{7} x_4 = \frac{59}{7}$
- $x_1 \quad +\frac{1}{7} x_3 + \frac{2}{7} x_4 = \frac{20}{7}$
- $x_2 \quad + x_4 = 3$  \hspace{1cm} (1)
- $-\frac{2}{7} x_3 + \frac{10}{7} x_4 + x_5 = \frac{23}{7}$

- Add CG cut from $x_1$ row (convenient to negate):
  - $-x_1 \geq -2$ \hspace{1cm} (a)

- Will want to isolate current basis. (1) + (a):
  - $\frac{1}{7}x_3 + \frac{2}{7}x_4 \geq \frac{6}{7}$ \hspace{1cm} (b)

- Add non-neg, integer excess var $x_6 \geq 0$:
  - $\frac{1}{7}x_3 + \frac{2}{7}x_4 - x_6 = \frac{6}{7}$ \hspace{1cm} (c)

- Note: $x_6$ integer because LHS-RHS in (a) is integer, also true for (b) since (1) is an equality.
• Add (c) and re-optimize.
• New optimal tableau:

\[
\begin{align*}
    z &= \frac{1}{2}x_5 + 3x_6 = \frac{15}{2} \\
    x_1 &= x_6 = 2 \\
    x_2 &= -\frac{1}{2}x_5 + x_6 = \frac{1}{2} \\
    x_3 &= -x_5 - 5x_6 = 1 \\
    x_4 &= \frac{1}{2}x_5 + 6x_6 = \frac{5}{2}
\end{align*}
\]

• \(B=\{1,2,3,4\}\) and \(x^*=(2, 1/2, 1, 5/2, 0, 0)\)

• Add another CG cut. Use a general rule to go from fractional tableau to a new cut.
General Rule for Deriving a Cut

• For row i with fractional RHS, the CG cut is\
\[ \sum_{j \in B'} (\bar{a}_{ij} - [\bar{a}_{ij}]) x_j \geq \bar{b}_i - [\bar{b}_i] \]

Example 1:
• \( z + \frac{4}{7} x_3 + \frac{1}{7} x_4 = \frac{59}{7} \)
  \( x_1 + \frac{1}{7} x_3 + \frac{2}{7} x_4 = \frac{20}{7} \)
  \( x_2 + x_4 = 3 \)
  \( -\frac{2}{7} x_3 + \frac{10}{7} x_4 + x_5 = \frac{23}{7} \)

• \( B = \{1, 2, 5\} \)
• Cut \( \frac{1}{7} x_3 + \frac{2}{7} x_4 \geq \frac{6}{7} \)

General Rule for Deriving a Cut

Example 2:
• \( Z + \frac{1}{2} x_5 + 3 x_6 = \frac{15}{2} \)
  \( x_1 + x_6 = 2 \)
  \( x_2 - \frac{1}{2} x_5 + x_6 = \frac{1}{2} \)
  \( x_3 - x_5 - 5 x_6 = 1 \)
  \( x_4 + \frac{1}{2} x_5 + 6 x_6 = \frac{5}{2} \)

• \( B = \{1, 2, 3, 4\} \)
• Cut \( \frac{1}{2} x_5 \geq \frac{1}{2} \) (\( \frac{1}{2} = -\frac{1}{2} - (\lfloor -\frac{1}{2} \rfloor) = -\frac{1}{2} - (-1) = \frac{1}{2} \))
• Add (c) and re-optimize.
• New optimal tableau:

\[
\begin{align*}
\text{z} & \quad + \frac{1}{2}x_5 + 3x_6 = 15/2 \\
x_1 & \quad + x_6 = 2 \\
x_2 & \quad - \frac{1}{2}x_5 + x_6 = \frac{1}{2} \\
x_3 & \quad - x_5 - 5x_6 = 1 \\
x_4 & \quad + \frac{1}{2}x_5 + 6x_6 = 5/2 \\
\end{align*}
\]

• \( B = \{1, 2, 3, 4\} \) and \( x^* = (2, 1/2, 1, 5/2, 0, 0) \)
• Add CG cut \( \frac{1}{2}x_5 \geq \frac{1}{2} \)
• Bring in integer, non-neg excess var \( x_7 \).

\[
\frac{1}{2}x_5 - x_7 = \frac{1}{2}
\]  

(d)

• Add (d) and re-optimize. New tableau:

\[
\begin{align*}
\text{z} & \quad + 3x_6 + x_7 = 7 \\
x_1 & \quad + x_6 = 2 \\
x_2 & \quad + x_6 - x_7 = 1 \\
x_3 & \quad - 5x_6 - 2x_7 = 2 \\
x_4 & \quad + 6x_6 + x_7 = 2 \\
x_5 & \quad - x_7 = 1 \\
\end{align*}
\]

• \( x^* = (2, 1, 2, 2, 1, 0, 0) \)

• Integral! \((x_1, x_2) = (2, 1)\) solves the original IP.

• Map the cuts back to the \((x_1,x_2)\) space
• First cut: \(x_1 \leq 2\)
• Second cut: \(\frac{1}{2}x_5 \geq \frac{1}{2}\). Substituting for \(x_5 = 3-2x_1 + 2x_2\), rearrange and obtain \(x_1 - x_2 \leq 1\).
Summary: Gomory’s algorithm

• Repeat:
  – Solve LP.
  – If integral, STOP. Else, generate a CG-cut from any row with fractional RHS.

• Thm. Gomory’s algorithm will solve an IP with integer coefficients and integer RHS values.

Gomory’s algorithm

• Not of practical interest, but transformative method!
• Can solve IPs without branch and bound, and inspired effort into identifying families of strong cuts.
• With some work, can extend to solve mixed IPs
Summary: Cutting plane method

• Cuts are valid inequalities that separate the current optimal, fractional solution
• The cutting-plane method
• Chvatal-Gomory cuts for IPs (they are complete!)
• Gomory’s cutting plane method
  – Solves IPs without branch and bound
  – Works on IPs with integer coefficients and integer RHS.