Section Notes 5
Review of Linear Programming

Applied Math / Engineering Sciences 121

Week of October 16, 2016

The following list of topics is an overview of the material that was covered in the lectures and sections. If anything is unclear or you have questions regarding a topic, ask your TF to clarify that issue. Proficiency in all of the topics listed below is necessary.

Contents

1 Lecture 2: Intro

- What is a linear program? (Section 1 - 2.1)
  A linear program (LP) is an optimization problem that involves maximizing or minimizing a linear objective function by choosing values for decision variables subject to a finite number of linear equality and inequality constraints
  - Proportionality: The contribution of each variable is directly proportional to its value in the objective function and constraints. If the value of a variable is doubled, so is its contribution.
  - Additivity: Contributions of variables into the objective function and constraints are via the sums (or differences) of individual contributions of each variable.
  - Certainty: The objective and constraint coefficients are data, that is, they are known con- stants.
- Standard Forms (Section 1 - 2.3)
- Putting things into canonical form (Section 2 - 2)
  - maximization
  - positive RHS
  - equality constraints
  - non-negative variables
  - isolated variables
- Matrix notation
- Graphical view (Section 1 - 2.2)
  - optimality
  - multiple optimal solutions
  - unbounded objective
  - infeasibility
- Graphical Interpretation of LPs
  - Be able to solve 2 variable linear programs using the graphical approach
  - Be able to use algebraic manipulation of the constraints to prove the optimality of graphically-found solutions
- Solutions for LPs (Section 2 - 1)
  - Basic Feasible Solutions: An LP in canonical form has an associated BFS where the isolated decision variables are non-zero and all other variables are zero.
  - When does it have a single optimal solution?
  - When does it have an infinite number of optimal solutions?

- When does it become infeasible?

2 Lecture 3: LP Modeling

- Modeling (Section 1 - 2.4)
  - Sets
  - Parameters
  - Variables
  - Understand the distinction and relationship between a genearlized mathematical model and the data from a particular problem instance.
- Finding ways to model non-linear aspects of a problem:
  - Minimize the maximum
  - Absolute value (Section 2 - Exercise 9)
  - Positive and negative parts (SailCo)
  - Specifying decision variables (Save-It)
  - Ratio constraints

3 Lecture 4: Convexity, Extreme points

- Convexity
- Polyhedron, hyperplane, halfspace
- Theorem: Polyhedron is a convex set
- Theorem: If $c^T x^* \geq c^T x'$ for all feasible extreme solutions x' that are adjacent to x^* then feasible, extreme solution x^* is optimal.
- Basic linear algebra: (Section 2 - 3.1)
  - A **basis** of R^m is a set of linearly independent m-dimensional vectors $v_1, ..., v_m$ with the property that every vector of R^m can be written as a linear combination of the vectors $v_1, ..., v_m$. Note that the vectors $v_1, ..., v_m$ form a square matrix that is invertible. These vectors $v_1, ..., v_m$ **span** the vector space R^m.
  - Basis of a matrix
    * A **basis** B for an arbitrary m-by-n matrix A can also be seen as a list of m numbers chosen from {1, 2,. . . , n} such that the square matrix A_B with m columns from A indexed by this list is a basis for R_m, i.e. the column vectors **span** the space R_m. Again, A_B will be invertible.
  - Linearly independent

    - Columns of matrix A are linearly independent if the only solution of $Ax = 0$ is $x = 0$
  - Span
    - These vectors $A_1, ..., A_m$ **span** the vector space R^m.
- Basic feasible solutions and extreme points (Section 2 - 3.1, 3.2)
  - **Basic variables** for a given basis B are the ones corresponding to the column vectors of B.
  - **Non-basic variables** for a given basis B are all variables except the **basic variables**.
  - The **basic solution x** of the system $Ax = b$ for a basis B is the unique solution of this system where all non-basic variables are equal to zero ($x_j = 0$ for all indices $j \notin B$).
  - A **feasible solution** is a solution for an LP which satisfies all the constraints.
  - A **basic feasible solution** is a solution for an LP that is both basic and feasible. Note that basic solutions for LPs in canonical form are solutions for $Ax = b$ but they might be infeasible if the non-negativity requirements for the decision variables are not satisfied.
  - Let S be the set of points in the feasible region of an LP. A point y in S is called an **extreme point** of S if y cannot be written as $y = \lambda w + (1 - \lambda)x$ for two distinct points w and x in S and $0 < \lambda < 1$. That is, y does not lie on the line segment joining any two points of S.
  - Every **basic feasible solution** of an LP corresponds to an **extreme point** of the feasible region of the LP.
- Theorem: Optimality of Extreme Points
  - Suppose P has at least 1 extreme point and there exists an optimal solution, then there exists an optimal extreme solution
- Matrix invertible $\Leftrightarrow$ columns span $\Leftrightarrow$ columns are linearly independent $\Leftrightarrow$ $Ax = b$ has unique solution
- Main LP assumptions (for an LP in canonical form)
  - Matrix A has full row rank
  - Matrix A has less rows than columns
  - Columns of A span
- Theorem: A matrix has a basis if and only if its columns span
- Theorem: Extreme points of polyhedron P are exactly the BFS.

4 Lecture 5: Primal Simplex

- Tableaux (Section 3 - 1.1)
  - Definition of simplex tableaux

- Getting from an LP to the Simplex Tableau
- Understand reduced costs, feasibility and optimality of the tableau (Section 3 - 5)

- Theorem: if B is a basis for A, there is tableaux corresponding to B
- Number of possible BFS
- Simplex method (Section 3 - 2)
  - Initialization (Phase I)
  - Check optimality
  - Choose entering index
  - Check unboundedness
  - Choose leaving index (Ratio Test)
  - Pivot to a new tableau, go to check optimality step
- Optimality condition
  - Reduced costs are all non-negative.
- Degeneracy (Section 3 - 3)
  - Degeneracy - At least one basic variable has value zero.
- Theorem: finite termination without degeneracy

5 Lecture 6: Advanced Primal Simplex

- Simplex Algorithm: Phase I (Section 3 - 4)
  - Auxiliary LP: Forcing artificial variables to 0
    - Theorem: The original LP is feasible if the auxiliary LP has an optimal solution of value 0
  - Process:
    - Introducing artificial variables
    - Find initial BFS
    - Use simplex to solve
    - Check for feasibility
      1. Optimal value is strictly positive - original problem is infeasible
      2. Optimal value is 0, and all artificial variables are non-basic - We have the initial feasible tableau for our original linear program
      3. Optimal value is 0, but artificial variables are basic - Degenerate tableau (See section note for discussion)

- Difference between slack and artificial variables
  - Slack variables transform inequalities into equalities. May take strictly positive or zero values at feasible solutions to original LP
  - Artificial variables are tools to find an initial feasible tableau. They must be zero at a feasible solution and are only used during Phase 1 to find a BFS for the original problem.
- Degeneracy
  - Degenerate basic solution
  - Degenerate tableau
  - When degeneracy occurs
- Cycling
  - Smallest subscript rule
  - Bland's theorem
  - General termination
- Fundamental theorems of LP
  - Theorem: LP is optimal, infeasible, or bounded
  - Lemma: If columns span, and an LP is feasible and not unbounded, then the LP has an optimal solution

6 Lecture 7: Duality

(Section 4 - 1)

- Formations:
  Given a program in standard equality form:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

  Our dual will have the form:

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & A^T y \geq c \\ & y \text{ free} \end{aligned}$$

- Converting between Primal and Dual

Primal	Dual
= Constraints	Free Variables
Non-negative Variables	$\geq$ constraints

The parallel formulations can be extended to LPs that are not in standard equality form. The following table specifies the equivalences between the primal and the dual in a general setting:

Primal (maximize)	Dual (minimize)
ith constraint $\leq$	ith variable ≥ 0
ith constraint $=$	ith variable free
ith constraint $\geq$	ith variable ≤ 0
jth variable ≤ 0	jth constraint $\leq$
jth variable free	jth constraint $=$
jth variable ≥ 0	jth constraint $\geq$

- Weak duality theorem
  - Primal Optimal Value $\leq$ Dual Optimal Value
- Corollary: If LP in standard equality form is unbounded then dual infeasible.
- Certificate of optimality
  - Given primal feasible solution x^* and dual feasible solution y^* where the primal and dual objectives are equal, then x^* is optimal for the primal problem and y^* is optimal for the dual problem
- Strong duality theorem (no gap)
  - If an LP has an optimal solution then so does its dual and the two optimal values are equal
- Lemma: optimal dual solution from primal
- Duality for general LPs

7 Lecture 8: Sensitivity Analysis

(Section 4 - 3)

- Sensitivity Analysis - robustness of solution
  Initial LP: $z - cT_B x_B - c_{B'} x_{B'} = 0$
  . $A_B xB + A_{B'} x_{B'} = b$
  Dual solution: $y^T = c_B^T A_B^{-1}$
  RHS: $\bar{b} = A_B^{-1} b$
  Reduced Costs: $\bar{c}_j = y^T A_j - c_j$
- Considering changes graphically
- Shadow price (for maximization problems): the amount by which the objective value is improved when the RHS value is increased by one unit
  - Shadow price $\equiv$ optimal dual value $\equiv$ reduced cost on slack variable in optimal primal tableau
- Generating AMPL sensitivity output

- Algebra: relating a tableau with basis B to the LP problem
- Different kinds of sensitivity changes:
  - Change objective coefficient of non-basic var: check reduced cost on changed coefficient
  - Change objective coefficient of basic var: check reduced cost on every nonbasic var
  - Change RHS: check $\bar{b} \geq 0$
  - Change entries in column of nonbasic var: check reduced cost of affected nonbasic var ("pricing out")
- Changing multiple parameters at once

8 Lecture 9: Complementary Slackness

- Theorem: need corresponding constraint to be tight whenever variable is strictly positive.
- In notation: $x_j e_j = 0$ and $y_i s_i = 0$ for primal slack variables s_i and dual excess variables e_j
- Checking optimality, restricted dual

9 Lecture 10: Dual Simplex

(Section 4 - 5)

- Naming convention (for understanding dual simplex)
- Motivation for dual simplex
- Tracking primal simplex in dual space
- Comparing Primal and Dual Simplex
- Dual simplex: dual pivots, track in primal
- Dual feasible (reduced-costs non-negative)
- Dual optimal (RHS non-negative)
- Dual simplex:
  - Assume have dual feasible tableau
  - Step 1: Pick basic variable, x_r, to leave with strictly negative RHS
  - Step 2: Pick nonbasic variable, x_k to enter by considering row r and strictly negative entries. Ratio test
  - Step 3: Pivot on (r,k) and go to step 1
  - Continue until have dual optimal tableau
- Why use dual simplex?

– Adding a new constraint

– Changing a RHS

– new Phase 1 method