AM 121: Intro to Optimization Models and Methods
Fall 2016

Lecture 7: LP Duality

David C. Parkes

Lesson Plan

- Certificate of optimality
- Primal and Dual LPs
- Weak duality theorem
- Strong duality theorem

Jensen & Bard: 4.2 before Complementary Slackness

Review

- **Standard equality form**

  max $c^T x$

  s.t. $Ax = b$

  $x \geq 0$

- **Standard inequality form**

  max $c^T x$

  s.t. $Ax \leq b$

  $x \geq 0$

LP Duality

- Linear programs come in **primal/dual pairs**

- Why is *LP duality* useful?
  - it leads to the **dual simplex** method, which has practical impact
  - it provides **new insight** into problems
  - it allows for **sensitivity analysis** (*next lecture*)
  - it provides a new way to reason about **optimality and infeasibility**

Establishing Optimality

- $\max\ x_1 - x_2 + 7x_3$
  s.t. $2x_1 - x_2 + x_3 = 1$
  $x_1 + x_2 + 2x_3 = 5$
  $x_1,\ x_2,\ x_3 \geq 0$

- Every feasible solution provides a lower bound of the optimal objective value

- Suppose find x=(0,1,2), value 13.
- Optimal val ≥ 13.

- *But how far away is x=(0,1,2) from optimality?*

Establishing optimality

- $\max\ x_1 - x_2 + 7x_3$ **(P)**
  s.t. $2x_1 - x_2 + x_3 = 1$ (a)
  $x_1 + x_2 + 2x_3 = 5$ (b)
  $x_1,\ x_2,\ x_3 \geq 0$

x=(0,1,2), value 13
Optimal?

- Example row operations:

$$\begin{array}{rrl} & 2x_1 - x_2 + x_3 = 1 & \text{(a)} \\ + & \underline{3(x_1 + x_2 + 2x_3) = 3(5)} & \text{3(b)} \\ & 5x_1 + 2x_2 + 7x_3 = 16 & \end{array}$$

$x_1 - x_2 + 7x_3 \leq 5x_1 + 2x_2 + 7x_3 = 16$

- Consider 3(a) + 2(b):

$$\begin{array}{rrl} & 3(2x_1 - x_2 + x_3) = 3(1) & \text{3(a)} \\ + & \underline{2(x_1 + x_2 + 2x_3) = 2(5)} & \text{2(b)} \\ & 8x_1 - x_2 + 7x_3 = 13 & \end{array}$$

$x_1 - x_2 + 7x_3 \leq 8x_1 - x_2 + 7x_3 = 13$

- Conclude x=(0,1,2) is **optimal**.

Certificate of Optimality

- Convince a friend that x is optimal by presenting "optimal constraint multipliers" $y^*=(3,2)$.
- Friend can follow these steps:
  - Verify x is feasible
  - Verify objective value x = upper bound implied by multipliers y*
- y^* provide a **certificate of optimality** for x^*
- **The dual LP finds these optimal multipliers**

- Adding constraints, **any** feasible solution x must satisfy:

$$y_1(2x_1 - x_2 + x_3) + y_2(x_1 + x_2 + 2x_3) = y_1 + 5y_2$$

$$\Rightarrow \quad (2y_1+y_2)x_1 + (-y_1 + y_2)x_2 + (y_1+ 2y_2)x_3 = y_1 + 5y_2$$

- For an *upper bound*, we need:
  $x_1 - x_2 + 7x_3 \le (2y_1+y_2)x_1 + (-y_1+y_2)x_2 + (y_1+2y_2)x_3 = y_1 + 5y_2$
- Since $x \ge 0$, sufficient $1 \le (2y_1 + y_2)$; $-1 \le (-y_1 + y_2)$; $7 \le (y_1 + 2y_2)$.

- Can solve:

$$\begin{aligned} \textbf{min} \quad & y_1 + 5y_2 \qquad \textbf{(D)} \\ \text{s.t.} \quad & 2y_1 + y_2 \ge 1 \\ & -y_1 + y_2 \ge -1 \\ & y_1 + 2y_2 \ge 7 \end{aligned}$$

- This is the **dual** for the original (**primal**) LP!
- By construction: optimal value of **(D)** ≥ optimal value of **(P).**
- In the example: $y^*=(3,2)$ is feasible for (**D**) with value 13; and $x^*=(0,1,2)$ is feasible for (**P**) with value 13.
- Conclude that y* optimal for (**D**)**;** x* optimal for (**P**).

Example: Primal and Dual LPs

- **max** $x_1 - x_2 + 7x_3$ (**P**)
  s.t. $2x_1 - x_2 + x_3 = 1$
  $x_1 + x_2 + 2x_3 = 5$
  $x_1, x_2, x_3 \geq 0$
- **min** $y_1 + 5y_2$ (**D**)
  s.t. $2y_1 + y_2 \geq 1$
  $-y_1 + y_2 \geq -1$
  $y_1 + 2y_2 \geq 7$

- *Going from the Primal to the Dual:*
  - coefficients primal objective → coefficients dual RHS
  - coefficients primal RHS → coefficients dual objective
  - dual "A matrix" is transpose of primal "A matrix"
    - # primal variables = # dual constraints
    - non-negative primal variables → "≥" dual constraints
    - # primal constraints = # dual variables
    - equality primal constraints → "free" dual variables

Primal/Dual for Standard Equality Form

- max $c^T x$ (**P**)
  s.t. $Ax = b$
  $x \geq 0$

- min $b^T y$ (**D**)
  s.t. $A^T y \geq c$

Primal	Dual
equality constraints	free variables
non-negative variables	≥ constraints

Example: Furniture problem (1 of 2)

- Make desks, tables and chairs.
- Resources: 48 lumber hours, 20 finishing hours, 8 carpentry hours.
- Desk profit $60, table $30 and chair $20.

	Desk	Table	Chair
Lumber	8	6	1
Finishing	4	2	1.5
Carpentry	2	1.5	0.5

Amount of each resource needed to make each type of furniture

Example: Furniture problem (2 of 2)

- max $z = 60x_1 + 30x_2 + 20x_3$

  s.t.
  $$8x_1 + 6x_2 + x_3 + x_4 = 48$$
  $$4x_1 + 2x_2 + 1.5x_3 + x_5 = 20$$
  $$2x_1 + 1.5x_2 + 0.5x_3 + x_6 = 8$$
  $$x_1, \ldots, x_6 \geq 0$$

  $x^* = (2,0,8,24,0,0)$, $z = 280$

- min $z = 48y_1 + 20y_2 + 8y_3$

  s.t.
  $$8y_1 + 4y_2 + 2y_3 \geq 60$$
  $$6y_1 + 2y_2 + 1.5y_3 \geq 30$$
  $$y_1 + 1.5y_2 + 0.5y_3 \geq 20$$
  $$y_1 \geq 0$$
  $$y_2 \geq 0$$
  $$y_3 \geq 0$$

  Constraints from A^Tb. Otherwise, y vars would be free.

  $y^* = (0,10,10)$, $z = 280$

Interpretation of Optimal Dual

- The optimal multiplier y_i is the **shadow price** on the *i*th constraint: the amount by which primal objective increases if RHS b_i increases by 1 (while the current basis remains optimal).
- In the furniture example, this represents the value of additional resources:
  - lumber = \$0
  - finishing = \$10
  - carpentry = \$10

Additional supply of lumber not useful (it is already in surplus in optimal sol!)

(Defer details until next lecture)

Constructing the Dual for a General LP

Primal

- $\max c^T x + d^T w$

  s.t. $Ax + Ew = b \quad (y_i)$

  $Fx + Gw \le e \quad (u_j)$

  $x \ge 0$, w free

Dual

- $\min b^T y + e^T u$

  s.t. $A^T y + F^T u \ge c \quad (x_k)$

  $E^T y + G^T u = d \quad (w_l)$

  y free, $u \ge 0$

Observations

$\begin{pmatrix} A & E \\ F & G \end{pmatrix}$ ➔ $\begin{pmatrix} A^T & F^T \\ E^T & G^T \end{pmatrix}$

obj coeff. ➔ RHS coeff.

equalities➔ free vars

inequal ➔ nonneg vars

free vars ➔ equalities

nonneg vars ➔ inequal

Example: Primal-Dual Pair

- max $x_1 - x_2$
  s.t. $-3x_1 + x_2 = 2$ $\quad$ y_1 free
  $2x_1 - x_2 \leq -1$ $\quad$ y_2 nonnegative
  $x_1 \geq 0$, x_2 free

- min $2y_1 - y_2$
  s.t. $-3y_1 + 2y_2 \geq 1$ $\quad$ x_1 nonnegative
  $y_1 - y_2 = -1$ $\quad$ x_2 free
  y_1 free, $y_2 \geq 0$

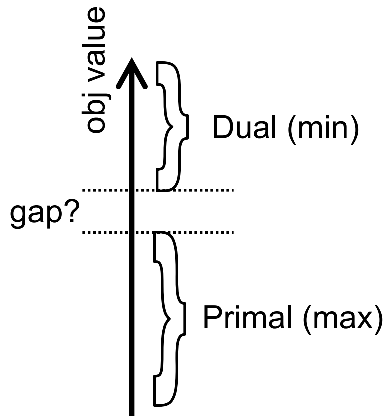
Primal/Dual for Standard Inequal. Form

- max $c^T x$ $\quad$ (P)
  s.t. $Ax \leq b$
  $x \geq 0$

- min $b^T y$ $\quad$ (D)
  s.t. $A^T y \geq c$
  $y \geq 0$

Obtain this by setting {d, A, E, G} to zero in recipe

Weak Duality Theorem



Theorem (weak duality): The objective value of any feasible primal solution is weakly less than the objective value of any feasible dual solution.

Weak Duality Theorem

- **Theorem.** The objective value of any feasible primal solution is weakly less than the objective value of any feasible dual solution.
- **Proof.** (prove for standard equality form)

max	$c^T x$	(P)
s.t.	$Ax = b$	
	$x \geq 0$	

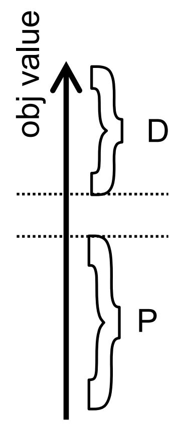
min	$b^T y$	(D)
s.t.	$A^T y \geq c$	

Consider x feasible for (P) and y feasible for (D).

$$c^T x \leq (A^T y)^T x = (y^T A) x = y^T A x = y^T b = b^T y$$

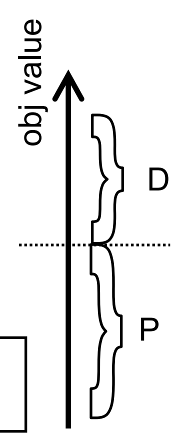
- **Corollary.** If a primal LP is unbounded then its dual problem is infeasible.
- **Proof**. Suppose for contradiction primal is unbounded and (D) has a feasible solution y. By weak duality, primal optimal value ≤ b^Ty. Contradiction.

- **Corollary**. If a dual LP is unbounded then its primal problem is infeasible.
- **Proof**. Suppose for contradiction dual is unbounded and (P) has a feasible solution x. By weak duality, dual optimal value ≥ c^Tx. Contradiction.



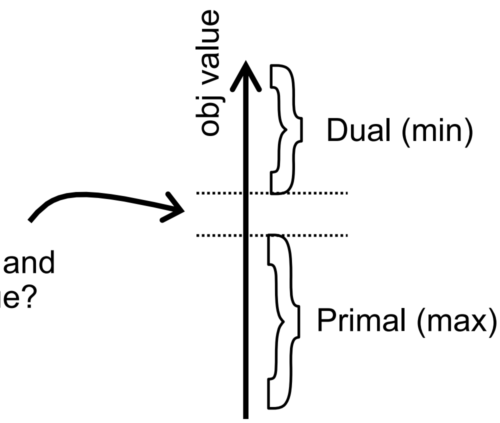
Certificate of Optimality

- **Corollary.** Given primal x* and dual y* with primal obj = dual obj, then x* is optimal for (P) and y* is optimal for (D).
- **Proof:**
- For any x, have $c^Tx \le b^Ty^*$ {weak duality} and thus $c^Tx \le c^Tx^*$ {$c^Tx^*=b^Ty^*$}.
- Thus: x^* is optimal for (P).
- Similar argument for y^* in (D).



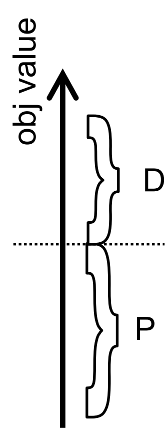
Can check optimality from primal and dual feasibility and comparing objective values.

will there ever
be a **gap** between
optimal dual value and
optimal primal value?



No! Strong Duality Theorem

- **Theorem (Strong Duality)**. If an LP has an optimal solution then so does its dual, and the optimal values are equal.



Review: Initial LP -> Tableau

- Construct tableau for basis B from initial LP:

  max z
  s.t. $z - c^T x = 0$
  $Ax = b, x \geq 0$

- Obtain (see Lecture 5):

$$z + c_B^T A_B^{-1} A_{B'} x_{B'} - c_{B'}^T x_{B'} = c_B^T A_B^{-1} b$$

$$I x_B + A_B^{-1} A_{B'} x_{B'} = A_B^{-1} b$$

Duality Lemma

- **Lemma**. If B is the **optimal** basis of a primal LP in standard equality form, then $\mathbf{y^T = c_B^T A_B^{-1}}$ is the optimal dual solution.

max	$c^T x$	(P)	min	$b^T y$	(D)
s.t.	$Ax = b$		s.t.	$A^T y \geq c$	
	$x \geq 0$				

- **Proof.** Let x* denote optimal primal solution. Let $\mathbf{y^T = c_B^T A_B^{-1}}$

  $z + (c_B^T A_B^{-1} A_{B'} - c_{B'}^T) x^*_{B'} = c_B^T A_B^{-1} b$, and so

  $z + (y^T A_{B'} - c_{B'}^T) x^*_{B'} = y^T b$ (*) {subst $\mathbf{y^T = c_B^T A_B^{-1}}$}

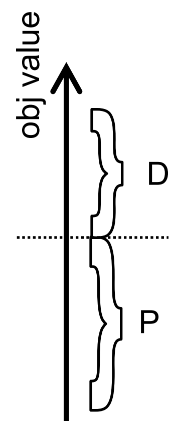
  Have $y^T A_{B'} - c_{B'}^T \geq 0$ because B is optimal, and so $A_{B'}^T y \geq c_{B'}$

  Since $A_B^T y = c_B$, have $A^T y \geq c$, and y is **dual feasible**.

  By (*): $c^T x^* = z = y^T b = b^T y$. By weak duality, since x^* is **optimal** for (P), then y is **optimal** for (D).

Strong Duality Theorem

- **Theorem (Strong Duality).** If an LP has an optimal solution then so does its dual, and the optimal values are equal.
- **Proof.** (for standard equality form)
  - Primal LP has an optimal solution, and thus an optimal BFS.
  - By duality lemma, the optimal dual solution has the same objective value as the optimal primal solution.



Example: Furniture problem

- max $z = 60x_1 + 30x_2 + 20x_3$

$$
\begin{aligned}
\text{s.t. } & 8x_1 + 6x_2 + x_3 + x_4 = 48 \\
& 4x_1 + 2x_2 + 1.5x_3 + x_5 = 20 \\
& 2x_1 + 1.5x_2 + 0.5x_3 + x_6 = 8 \\
& x_1, \ldots, x_6 \geq 0
\end{aligned}
$$

$x^* = (2,0,8,24,0,0)$, $z = 280$

- min $z = 48y_1 + 20y_2 + 8y_3$

$$
\begin{aligned}
\text{s.t. } & 8y_1 + 4y_2 + 2y_3 \geq 60 \\
& 6y_1 + 2y_2 + 1.5y_3 \geq 30 \\
& y_1 + 1.5y_2 + 0.5y_3 \geq 20 \\
& y_1 \geq 0 \\
& y_2 \geq 0 \\
& y_3 \geq 0
\end{aligned}
$$

$y^* = (0,10,10)$, $z = 280$

Example: Furniture problem

- **Optimal (primal) tableau:**

$$
\begin{aligned}
z \quad +5x_2 \qquad\qquad +10x_5 + 10x_6 &= 280 \\
-2x_2 \qquad + 1x_4 + 2x_5 - 8x_6 &= 24 \\
-2x_2 + x_3 + \qquad 2x_5 - 4x_6 &= 8 \\
x_1 + 1.25x_2 \qquad -0.5x_5 + 1.5x_6 &= 2
\end{aligned}
$$

$B=\{4,3,1\}$
$x^*=(2,0,8,24,0,0)$,
$z=280$

- Dual solution: $\mathbf{y^T=c_B{}^TA_B{}^{-1}}$

Note 1: The order of columns in defining A_B and c_B is 4, 3, 1 because row 1 in the final primal tableau isolates x4, row 2 isolates x3 and row 3 isolates x1.

- $A_B=\begin{pmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{pmatrix}$ $\quad A_B{}^{-1}=\begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix}$

Note 2: The values of $A_B{}^{-1}$ can also be read directly from final tableau (from cols. on INITIAL isolated variables). Optimal dual can also be read from final tableau.

- $(y_1,y_2,y_3)^T=(0, 20, 60)^T\begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix} = (0, 10, 10)^T$

Reading Dual from Final Tableau

- The final primal tableau also gives optimal dual sol:

$$z \qquad +5x_2 \qquad \mathbf{+0}x_4 \mathbf{+10}x_5 \mathbf{+10}x_6 = 280$$

- If xj is basic in the initial tableau then reduced cost cj in final tableau gives dual value yj corresponding to the primal equality
- Read off $(y_1,y_2,y_3)^T=$ $(0, 10, 10)^T$ from the "z equation"

- *Careful* : read the coefficient for corresponding slack variable, … they may be out of order. For example, if final tableau is:

$$z \qquad +5x_2 \qquad \mathbf{+10}x_5 \mathbf{+0}x_4 \mathbf{+10}x_6 = 280$$

then we'd still read optimal dual as $(y_1,y_2,y_3)^T=$ $(0, 10, 10)^T$.

Why can dual be read off final tableau?

- Given an optimal (primal) basis B, then:
  RHS: $\mathbf{\bar{b}}=\mathbf{A_B^{-1}b}$
  Dual solution: $y^T = c_B^T A_B^{-1}$
  Objective value: $z = c_B^T A_B^{-1} b = y^T b$
  Nonbasic obj coeff: $\bar{c_{B'}}^T = (c_B^T A_B^{-1} A_{B'} - c_{B'}^T)$
  For nonbasic j, $\mathbf{\bar{c}_j = c_B^T A_B^{-1} A_j - c_j = y^T A_j - c_j}$

- Immediate observations:
  (a) $\bar{c}_j = y_j$ for slack variable x_j since $c_j = 0$ and $A_j = e_j$ (i.e., the j^{th} unit vector) ← can read optimal dual directly from optimal primal tableau
  (b) dual variable y_j is shadow price on RHS of constraint j (since $z = y^T b$) ← the optimal dual value gives "shadow price"

Why can A_B^{-1} be read off final tableau?

Let $\tilde{B}$ and $\tilde{B}'$ denote the isolated and non-isolated variables in the initial tableau.

We have

$$A_{\tilde{B}} x_{\tilde{B}} + A_{\tilde{B}'} x_{\tilde{B}'} = b, \tag{1}$$

with $I = x_{\tilde{B}}$ by arranging columns according to order of $\tilde{B}$.

To get to the tableau for the optimal basis B, we multiply (1) by A_B^{-1}, to obtain

$$\bar{A}x = A_B^{-1} A_{\tilde{B}} x_{\tilde{B}} + A_B^{-1} A_{\tilde{B}'} x_{\tilde{B}'} = A_B^{-1} b = \bar{b} \tag{2}$$

In particular, since $I = A_{\tilde{B}}$ this means we can read off A_B^{-1} from the coefficients $\bar{A}_{\tilde{B}}$ $(= A_B^{-1} A_{\tilde{B}} = A_B^{-1})$ in the final tableau that correspond to the columns of the initial isolated variables $\tilde{B}$.

- **Theorem.** The dual of the dual of an LP is the original LP.
- Can establish this knowing nothing more than the rules for duality. Exercise.

Summary: Duality

(f) (b) (e)

PRIMAL\DUAL	infeasible	has optimal solution	unbounded
infeasible	possible	impossible	possible
has optimal solution	impossible	possible	impossible
unbounded	possible	impossible	impossible

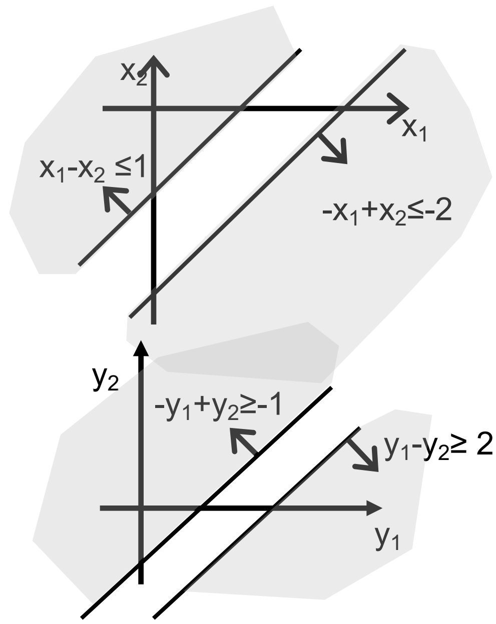
(d) (c) (a)

(a) consequence of weak duality
(b) consequence of strong duality
(c) e.g., a primal LP with an optimal solution
(d) e.g., an unbounded primal LP
(e) e.g., an unbounded dual LP
(f)

Primal and Dual Infeasible

- $\max 2x_1 - x_2$
  s.t. $x_1 - x_2 \leq 1$
  $-x_1 + x_2 \leq -2$
  $x_1, \quad x_2 \geq 0$

- $\min y_1 - 2y_2$
  s.t. $y_1 - \quad y_2 \geq 2$
  $-y_1 + \quad y_2 \geq -1$
  $y_1, \quad y_2 \geq 0$



Summary: LP Duality

• Every primal problem has a dual problem, and the dual of the dual is the primal problem

• Optimal dual solutions provide economic intuition (and allow sensitivity analysis, next lecture).

•**Weak duality theorem**: feasible primal value ≤ feasible dual value

- Certificate of optimality
- Primal unbounded => dual infeasible

•**Strong duality thm**: if LP has solution, then optimal primal value = optimal dual value