AM 121: Intro to Optimization Models and Methods
Fall 2016

Lecture 6: Phase I, degeneracy, smallest subscript rule.

David C. Parkes

Lesson Plan

- Review: simplex method, proof of termination
- Phase 1 (initialization)
- Degeneracy, cycling, smallest subscript rule.
- The **Fundamental Theorem of Linear Programming.**

Textbook Readings: 3.7 and 3.8

Review: A Tableau

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & z - c^T x = 0 \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

- **Definition.** The **tableau** for basis B is a system of eqns where the **basic variables are isolated**.

- For basis B (with $B' = N \setminus B$) the **tableau** is:

$$\begin{aligned} z \qquad & + \bar{c}_{B'}^T x_{B'} = \bar{v} \\ I x_B & + \bar{A}_{B'} x_{B'} = \bar{b} \end{aligned}$$

Example

- max $z = x_1 + x_2$
- s.t. $x_1 \leq 2$

  $x_1 + 2x_2 \leq 4$

  $x_1, \; x_2 \geq 0$

- max $z = x_1 + x_2$
- s.t. $x_1 + x_3 = 2$

  $x_1 + 2x_2 + x_4 = 4$

  $x_1, \; x_2, \; x_3, \; x_4 \geq 0$

- Initial tableau (for basis {3,4}):

$$\begin{aligned} z - x_1 - x_2 &= 0 \\ x_1 + x_3 &= 2 \\ x_1 + 2x_2 + x_4 &= 4 \end{aligned}$$

Example of Simplex Method

$$\begin{aligned} z - x_1 - x_2 &= 0 \\ \leftarrow \quad \boxed{x_1} + x_3 &= 2 \\ x_1 + 2x_2 + x_4 &= 4 \\ \uparrow \quad & \end{aligned}$$

Basic x_3 x_4
Ratio 2/1 4/1
x_1 to enter. x_3 to leave. pivot(3,1)

$$\begin{aligned} z - x_2 + x_3 &= 2 \\ x_1 + x_3 &= 2 \\ \leftarrow \quad \boxed{2\,x_2} - x_3 + x_4 &= 2 \\ \uparrow \quad & \end{aligned}$$

Basic x_1 x_4
Ratio 2/2
x_2 to enter. x_4 to leave. pivot(4,2)

$$\begin{aligned} z + \tfrac{1}{2} x_3 + \tfrac{1}{2} x_4 &= 3 \\ x_1 + x_3 &= 2 \\ x_2 - \tfrac{1}{2} x_3 + \tfrac{1}{2} x_4 &= 1 \end{aligned}$$

Basic x_1 x_2

Reduced costs all nonnegative.
Stop!

- Solution: $(x_1,x_2,x_3,x_4)=(2,1,0,0)$, $z=3$.

Comments

- 1. We need to be able to find an initial tableau corresponding to a BFS
- 2. $\bar{c}_k$ is the **reduced cost** of nonbasic variable x_k. Amount by which z *decreases* when x_k increases (and so $\bar{c}_k<0$ is good).

3. Unboundedness

- $x_i + \sum_{j \in B'} \bar{a}_{ij} x_j = \bar{b}_i$ (for all $i \in B$)
- Because other nonbasic vars = 0, we can increase x_k while:

  $x_i = \bar{b}_i - \bar{a}_{ik} x_k \geq 0$ (for all $i \in B$)
- If $\bar{a}_{ik} \leq 0$ for every i in B, then x_k can increase without bound (without affecting objective)!

4. Pivoting to the new Tableau

- **Definition**. A **pivot** on (r,k) is row operations to construct tableau for $B := B \cup \{k\} \setminus \{r\}$.
- **(a)** Divide row $x_r + \sum_{j \in B'} \bar{a}_{rj} x_r = \bar{b}_r$ through by $\bar{a}_{rk}$ so that coefficient of new basic variable x_k becomes 1.

  (Why does RHS of row r remain nonnegative?)

  A: the coefficient $\bar{a}_{rk}$ is strictly positive!

- **(b)** Add/subtract multiples of this adjusted row to all other equations (including objective) to remove x_k

  (Why do these operations not affect isolation of other basic vars?)

  A: the only basic variable with non-zero coefficient in row r is x_r

  (Why does the RHS of the other rows remain nonnegative?)

  A: for a row r' with positive coefficient $a_{r'k}$ we subtract multiple $\bar{a}_{r'k}/\bar{a}_{rk}$ of row r, and $(\bar{a}_{r'k}/\bar{a}_{rk})\bar{b}_r \leq \bar{b}_{r'k}$ by the ratio test.

[Note: we're doing "Gauss-Jordan elimination."]

Degeneracy

- A BFS is **degenerate** if a basic variable x_i has value zero.
- *Ratio test*. $t^* = \min\{ \bar{b}_i/a_{ik} : i \in B, \bar{a}_{ik} > 0\}$. Pick leaving index $r \in B$ with min ratio.
- If $\bar{a}_{ik} > 0$ and $\bar{b}_i = 0$, then simplex method cannot make the entering variable x_k increase in value.

- Move to an adjacent basis, but without improving objective.
- Ignore this possibility for a moment.

Simplex Termination

- **Theorem.** Simplex method terminates with an optimal solution, or a proof of unboundedness, as long as never reaches a degenerate BFS.
- **Proof.** Suppose LP is **not** unbounded.
  - In every iteration the value of the entering variable $x_k := t^* > 0$, and objective **strictly** increases.
    => cannot visit same BFS twice.
    => terminates, since finite number of BFS.

  - **If unbounded***:* must reach a tableau that is adjacent to one in which can increase objective without bound.

Remaining Issues

- How to find a first BFS to initialize the simplex method?
- How can we be sure the simplex method will terminate even if there may be degenerate BFSs?

Finding an initial BFS

- **Easy case**: If our initial LP in standard inequal. form

$$\begin{aligned} \max\ & c^T x \\ \text{s.t.}\ & Ax \leq b \\ & x \geq 0 \end{aligned}$$

and $b \geq 0$, can transform into canonical form by introducing slack variables.

- Example:

$$\begin{aligned} \max\ & x_1 + x_2 \\ \text{s.t.}\ & x_1 \leq 2 \\ & x_1 + 2x_2 \leq 4 \\ & x_1,\ x_2 \geq 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} z - x_1 - x_2 \qquad\qquad &= 0 \\ x_1 \qquad + x_3 \qquad &= 2 \\ x_1 + 2x_2 \qquad + x_4 &= 4 \end{aligned}$$

Initialization: General case

- **LP with +ve RHS**, but may have ≥ and = constraints

$$\begin{array}{ll} \max & 2x_1 + x_2 \\ \text{s.t.} & x_1 + x_2 \le 3 \\ & -x_1 + x_2 \ge 1 \\ & x_1, \; x_2 \ge 0 \end{array}$$

$$\begin{array}{ll} \max & 2x_1 + x_2 \\ \text{s.t.} & x_1 + x_2 + x_3 = 3 \\ & -x_1 + x_2 - x_4 = 1 \\ & x_1, \; x_2, \; x_3, \; x_4 \ge 0 \end{array} \qquad (1)$$

- Don't have a basis. Not even sure if feasible!
- Introduce "**artificial variable**" $x_5 \ge 0$.

  $-x_1 + x_2 - x_4 + x_5 = 1$
- **Auxiliary LP:**

$$\begin{array}{ll} \min & x_5 \\ \text{s.t.} & x_1 + x_2 + x_3 = 3 \\ & -x_1 + x_2 - x_4 + x_5 = 1 \\ & x_1, \; x_2, \; x_3, \; x_4, \; x_5 \ge 0 \end{array} \qquad (2)$$

- **Lemma.** (1) feasible iff (2) has optimal soln with $x_5=0$

  (➔) can set $x_5=0$ in (2)

  (🡐) if opt soln with $x_5=0$, then $x_1 \ldots x_4$ feasible for (1)

Phase 1 of the simplex method

- Introduce **artificial variables** in "≥" and "=" rows. Solve auxiliary problem to check feasibility
-

$$\begin{array}{lll} \max & w & \\ \text{s.t.} & w + \mathbf{x_5} = 0 & \text{(a)} \\ & x_1 + x_2 + \mathbf{x_3} = 3 & \text{(b)} \\ & -x_1 + x_2 - x_4 + \mathbf{x_5} = 1 & \text{(c)} \\ & x_1, \ldots, x_5 \ge 0 & \end{array}$$

- **Why did this help?** Easy BFS for **auxiliary LP**!
- $\mathbf{x_3}$ but not $\mathbf{x_5}$ isolated. To isolate $\mathbf{x_5}$ can use (a) - (c).
- Get tableau for B={3,5}:

$$\begin{array}{l} w + x_1 - x_2 + x_4 = -1 \\ x_1 + x_2 + x_3 = 3 \\ -x_1 + x_2 - x_4 + x_5 = 1 \end{array}$$

- Can now solve with simplex. If obtain w=0, **can find an initial BFS for original problem.**

Phase 1-Phase 2 Example (1 of 2)

$$\begin{array}{llr} w + x_1 - x_2 + x_4 = -1 & \text{(a)} \\ x_1 + x_2 + x_3 = 3 & \text{(b)} \\ \leftarrow\ -x_1 + \boxed{x_2} - x_4 + x_5 = 1 & \text{(c)} \end{array}$$

Basic x_3 x_5
Ratio 3/1 1/1
x_2 to enter, x_5 to leave

$$\begin{array}{ll} w + x_5 = 0 & (*) \\ 2x_1 + x_3 + x_4 - x_5 = 2 \\ -x_1 + x_2 - x_4 + x_5 = 1 \end{array}$$

B={2,3}. Optimal.
x=(0,1,2,0,0); w=0

- **Can we find a BFS for original LP?**
- Drop (*) and x_5 (since x_5=0), and obtain system:

$$\begin{array}{l} 2x_1 + x_3 + x_4 = 2 \\ -x_1 + x_2 - x_4 = 1 \end{array}$$

As long as final BFS is non-degenerate, x5 (=0) will be non-basic and we have a basis for the original LP ({2,3}).

Phase 1-Phase 2 Example (2 of 2)

$$\begin{array}{ll} z - 2x_1 - \mathbf{x_2} = 0 & \text{(a)} \leftarrow \text{original obj} \\ 2x_1 + \mathbf{x_3} + x_4 = 2 & \text{(b)} \\ -x_1 + \mathbf{x_2} - x_4 = 1 & \text{(c)} \end{array}$$

- Need to isolate $\{x_2,x_3\}$. Do (a) + (c). **Now begin Phase 2**.

$$\begin{array}{l} z - 3x_1 - x_4 = 1 \\ \leftarrow\ \boxed{2x_1} + x_3 + x_4 = 2 \\ -x_1 + x_2 - x_4 = 1 \end{array}$$

Basic x_2 x_3
Ratio 2/2
Pick x_1 to enter. x_3 leaves.

$$\begin{array}{l} z + \tfrac{1}{2} x_3 + \tfrac{1}{2} x_4 = 4 \\ x_1 + \tfrac{1}{2} x_3 + \tfrac{1}{2} x_4 = 1 \\ x_2 + \tfrac{1}{2} x_3 - \tfrac{1}{2} x_4 = 2 \end{array}$$

B={1,2}.
Optimal. x=(1,2,0,0).
z=4

Summary: Phase 1

- Introduce **artificial variables** in "≥" and "=" rows
- Solve **auxiliary LP** to find solution with all artificial variables taking on value zero
- If exists, then this solution **provides a BFS for the original LP**. Else, original LP is infeasible.
- A key property of the auxiliary LP is that it has a BFS that is easy to identify.

Degeneracy

- **Definition.** A **basic solution** is **degenerate** when one or more basic variables have value zero.
- **Definition.** A **tableau** is **degenerate** when one or more RHS values $\bar{b}_i$ have value zero

- Whenever we have to choose between several leaving indices, the next tableau is degenerate…

Example (Degeneracy)

- $$\begin{aligned} \max\ & 2x_1 + x_2 \\ \text{s.t.}\ & x_1 - x_2 \le 1 \\ & x_1 \le 1 \\ & x_2 \le 1 \\ & x_1,\ x_2 \ge 0 \end{aligned}$$

- Initial tableau (non degenerate):

$$\begin{aligned} z - 2x_1 - x_2 &= 0 \\ x_1 - x_2 + x_3 &= 1 \\ x_1 + x_4 &= 1 \\ x_2 + x_5 &= 1 \end{aligned}$$

B={3,4,5}
x=(0,0,1,1,1)

Example (Degeneracy)

$$\begin{aligned} z - 2x_1 - x_2 &= 0 \\ \boxed{x_1} - x_2 + x_3 &= 1 \\ x_1 + x_4 &= 1 \\ x_2 + x_5 &= 1 \end{aligned}$$

Basic x_3, x_4, x_5
Ratio **1/1** **1/1**
x_1 to enter, x_3 to leave (tie break)

$$\begin{aligned} z - 3x_2 + 2x_3 &= 2 \\ x_1 - x_2 + x_3 &= 1 \\ \boxed{x_2} - x_3 + x_4 &= 0 \\ x_2 + x_5 &= 1 \end{aligned}$$

Basic x_1, x_4, x_5
Ratio 0/1 1/1
x=(1,0,0,0,1)
x_4=0. **degenerate**!
x_2 to enter, x_4 to leave

$$\begin{aligned} z - x_3 + 3x_4 &= 2 \\ x_1 + x_4 &= 1 \\ x_2 - x_3 + x_4 &= 0 \\ \boxed{x_3} - x_4 + x_5 &= 1 \end{aligned}$$

Basic x_1, x_2, x_5
Ratio 1/1
x=(1,0,0,0,1)
x_2=0. **degenerate**!
x_3 to enter, x_5 to leave

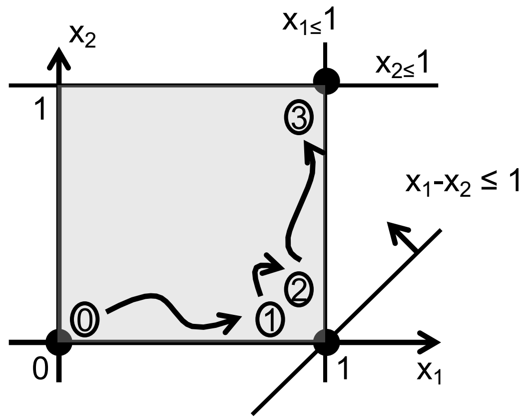
Basis and tableau has changed, but BFS and obj value unchanged

$$\begin{aligned} z + 2x_4 + x_5 &= 3 \\ x_1 + x_4 &= 1 \\ x_2 + x_5 &= 1 \\ x_3 - x_4 + x_5 &= 1 \end{aligned}$$

Basic x_1, x_2, x_3
x=(1,1,1,0,0)

Optimal solution! Was OK here ☺

Non basic	Binding
x_1, x_2	$x_1=0, x_2=0$
x_2, x_3	$x_2=0, x_1-x_2=1$
x_3, x_4	$x_1-x_2=1, x_1=1$
x_4, x_5	$x_1=1, x_2=1$



- 5 vars, 3 equations. Each basic solution adds n-m=2 additional binding constraints (nonbasic vars = 0), implies unique solution.
- Degeneracy occurs when more than (n-m) constraints intersect at an extreme point (e.g., point (1,0).)

Degeneracy and Cycling

- Will simplex method terminate?
- Objective value does not strictly increase at each iteration. Earlier proof fails.

- **Definition.** The simplex method **cycles** when it returns to the same tableau
  - E.g., T_0 ➔ T_1 ➔ … ➔ T_{p-1} ➔ T_0

- In this case, simplex method would **cycle** forever!

From “method” to algorithm:

- Need to make precise remaining design choices

- Choice of entering index:
  - **most negative reduced cost**: choose $k \in B'$ with smallest $\bar{c}_k$
  - **smallest subscript**: choose smallest index $k \in B'$ with $\bar{c}_k < 0$
  - **random:** choose any $k \in B'$ with $\bar{c}_k < 0$

- Choice of leaving index: (may be a tie)
  - **smallest subscript**: choose smallest index $r \in R$
  - **random:** choose any $r \in R$

A Bad Rule

1. Pick the entering variable with the most negative reduced cost (break ties according to index)
2. Pick the exiting variable with the smallest index

Cycling example

(Chvatal ’83)

- $\max\ 10x_1 - 57x_2 - 9x_3 - 24x_4$
  s.t. $0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 \le 0$
  $0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 \le 0$
  $x_1 \le 1$
  $x_1, x_2, x_3, x_4 \ge 0$

optimal solution
$x=(1,0,1,0)$, value 1

- Put in canonical form:

$$
\begin{array}{lr}
z - 10x_1 + 57x_2 + 9x_3 + 24x_4 & = 0 \\
\leftarrow\ \boxed{0.5x_1} - 5.5x_2 - 2.5x_3 + 9x_4 + x_5 & = 0 \\
0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + x_6 & = 0 \\
x_1 + x_7 & = 1 \\
\uparrow &
\end{array}
$$

x_5	x_6	x_7
0/0.5	0/0.5	1/1

$$
\begin{array}{lr}
z - 53x_2 - 41x_3 + 204x_4 + 20x_5 & = 0 \\
x_1 - 11x_2 - 5x_3 + 18x_4 + 2x_5 & = 0 \\
\leftarrow\ \boxed{4x_2} + 2x_3 - 8x_4 - x_5 + x_6 & = 0 \\
11x_2 + 5x_3 - 18x_4 - 2x_5 + x_7 & = 1 \\
\uparrow &
\end{array}
$$

x_1	x_6	x_7
	0/4	1/11

$$
\begin{array}{lr}
z - 14.5x_3 + 98x_4 + 6.75x_5 + 13.25x_6 & = 0 \\
x_1 + \boxed{0.5x_3} - 4x_4 - 0.75x_5 + 2.75x_6 & = 0 \\
x_2 + 0.5x_3 - 2x_4 - 0.25x_5 + 0.25x_6 & = 0 \\
-0.5x_3 + 4x_4 + 0.75x_5 - 2.75x_6 + x_7 & = 1
\end{array}
$$

x_1	x_2	x_7
0/.5	0/.5	

$$
\begin{array}{lr}
z + 29x_1 - 18x_4 - 15x_5 + 93x_6 & = 0 \\
2x_1 + x_3 - 8x_4 - 1.5x_5 + 5.5x_6 & = 0 \\
-x_1 + x_2 + \boxed{2x_4} + 0.5x_5 - 2.5x_6 & = 0 \\
x_1 + x_7 & = 1
\end{array}
$$

x_3	x_2	x_7
	0/2	

$$
\begin{array}{lr}
z + 20x_1 + 9x_2 - 10.5x_5 + 70.5x_6 & = 0 \\
-2x_1 + 4x_2 + x_3 + \boxed{0.5x_5} - 4.5x_6 & = 0 \\
-0.5x_1 + 0.5x_2 + x_4 + 0.25x_5 - 1.25x_6 & = 0 \\
x_1 + x_7 & = 1
\end{array}
$$

x_3	x_4	x_7
0/.5	0/.25	

$$
\begin{array}{lr}
z - 22x_1 + 93x_2 + 21x_3 - 24x_6 & = 0 \\
-4x_1 + 8x_2 + 2x_3 + x_5 - 9x_6 & = 0 \\
+0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + \boxed{x_6} & = 0 \\
x_1 + x_7 & = 1
\end{array}
$$

x_5	x_4	x_7
	0/1	

$$
\begin{array}{lr}
z - 10x_1 + 57x_2 + 9x_3 + 24x_4 & = 0 \\
0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_5 & = 0 \\
0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + x_6 & = 0 \\
x_1 + x_7 & = 1
\end{array}
$$

x_5	x_6	x_7

where we started!!

Smallest subscript rule

- Entering: amongst those with strictly negative reduced cost, pick var with smallest index.
- Exiting: amongst those with min ratio, pick variable with smallest index.

- **Bland's Theorem**. If the simplex method uses the smallest subscript rule then it will terminate.
- Proof: See Chvatal "Linear Programming" 1983

Fundamental Theorem of LP

- **Theorem**. Any LP has either an optimal solution, is infeasible, or is unbounded.
- **Proof** (sketch)**:**
  - Case 1: Feasible. (Ok!)
  - Case 2: Unbounded. (Ok!).
  - Case 3: Feasible and bounded. Convert to standard equality form. Appeal to simplex lemma.

- *Note*: some optimization problems do not have this property, e.g. min $1/x$ s.t. $x \geq 1$

Simplex lemma

- Consider an LP in standard equality form (max c^Tx s.t. $Ax=b$, $x\geq 0$) with columns of A that span.

- **Lemma**. If an LP in standard equal. form is feasible and bounded, then it has an optimal solution.
- **Proof**. (*sketch*)
  - If **feasible** then LP has a BFS (use Phase 1 with smallest subscript rule; must terminate with optimal value zero.)
  - Obtain BFS for original LP from final tableau of Phase 1 (need columns of A to span for this)
  - Simplex with smallest subscript rule for Phase 2. Must terminate. Since **not unbounded**, must terminate with optimal solution.

Comments on Optimality

- Consider a BFS x.
- If the reduced costs are non-negative, then x is optimal. This is true whether or not x is degenerate. Thus, it is a sufficient test.
- If x is optimal and nondegenerate then the reduced costs will be non-negative. But, a degenerate BFS x can be optimal with negative reduced costs!
- There is no simple test for determining whether a degenerate BFS is optimal.
- The simple test of non-negative reduced costs is sufficient for the simplex method: Bland's theorem tells us that this optimality test ensures termination, even in the presence of degeneracy.

Comments on Unique Optimality

- Consider a BFS x.
- If the reduced costs are positive, then x is the unique optimal solution. This is true whether or not x is degenerate. It is a sufficient test.
- If BFS x is optimal and nondegenerate then the reduced costs will be positive.
- But, a degenerate BFS x can be the unique optimal solution but have non-positive reduced costs.
- There is no simple test for determining whether a degenerate BFS is unique optimal.

Handling Degeneracy in Phase 1?

(Advanced topic)

- Phase 1 must terminate with non-degenerate basic solution to be able to construct BFS for original LP
- Phase 1 may terminate with artificial variable $u_i=0$, but basic. Suppose equation is $\sum_{j=1}^{n} \bar{a}_{ij}x_j + u_i=0$
- If $\bar{a}_{ij}=0$ for all j then can delete entire equation (redundant constraint)
- Else, some $\bar{a}_{ij}\neq 0$. Pivot on entry (i,j), cause x_j to become basic and variable u_i to become nonbasic.
- Repeat this process until all artificial variables are "driven out" of the (phase 1) basis.

Summary: Simplex method

- **Phase 1 (auxiliary LP)** can be formulated to find an initial BFS
- Degeneracy (basic variables taking on value zero) occurs when more than *n-m* constraints intersect on a feasible point
- Cycling can be prevented through the **smallest subscript rule**.
- **Fundamental thm. of LP**: Every LP has an opt. solution, is infeasible, or is unbounded.