

AM 121: Intro to Optimization Models and Methods Fall 2016

Lecture 5: The Simplex Method



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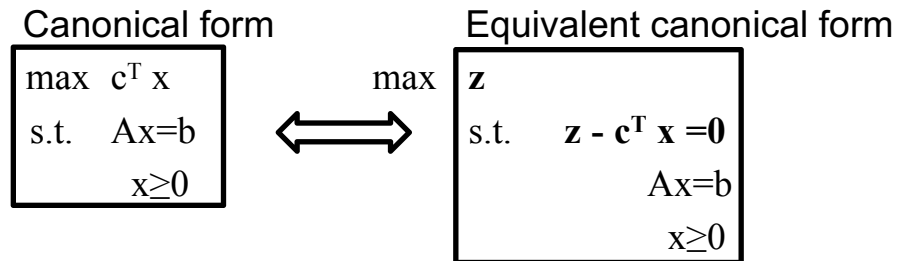
Lesson Plan

This lecture: Moving towards an algorithm for solving LPs

- Tableau. Adjacent BFS.
- The **simplex method**

- A first theorem about the simplex method
Jensen & Bard: 3.4, 3.5, 3.6,
3.10 (for matrix representation)

Warm-up



- We're interested in constructing a **tableau** that corresponds to a basis B
- A tableau is a system of equations
- Represents **everything** about an LP

Example

- $$\begin{array}{l} \max \quad x_1 + x_2 \\ \text{s.t.} \quad x_1 \quad \quad \quad + x_3 = 2 \\ \quad \quad x_1 + 2x_2 \quad \quad + x_4 = 4 \\ \quad \quad x_1, \quad x_2, \quad x_3, \quad x_4 \geq 0 \end{array}$$
- $$\begin{array}{l} \max \quad z \\ \text{s.t.} \quad z - x_1 - x_2 = 0 \\ \quad \quad x_1 \quad \quad \quad + x_3 = 2 \\ \quad \quad x_1 + 2x_2 \quad \quad + x_4 = 4 \\ \quad \quad x_1, \quad x_2, \quad x_3, \quad x_4 \geq 0 \end{array}$$

- The **tableau** corresponding to basis {3,4} is:

$$\begin{array}{l} z - x_1 - x_2 = 0 \\ x_1 \quad \quad + x_3 = 2 \\ x_1 + 2x_2 \quad + x_4 = 4 \end{array} \quad \text{(everything about LP!)}$$

The Tableau

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & z - c^T x = 0 \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

• **Definition.** The **tableau** for basis B is a system of eqns where the **basic variables are isolated**.

• For basis B (with $B' = N \setminus B$) the **tableau** is:

$$\begin{aligned} z \quad & + \bar{c}_{B'}^T x_{B'} = \bar{v} \\ Ix_B + \bar{A}_{B'} x_{B'} & = \bar{b} \end{aligned}$$

- Basic solution $x_B = \bar{b}, x_{B'} = 0$.
- Objective $z = \bar{v}$
- Feasible when $\bar{b} \geq 0$
- Parameters $\bar{A}, \bar{c}, \bar{v}, \bar{b}$ different from those without bar
- RHS may be negative

Insisting on $x_{B'} = 0$ adds an additional $n-m$ constraints. Pins down a solution to the LP!

Constructing a Tableau

• Construct the tableau for basis B :

$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & z - c^T x = 0 \\ & Ax = b \\ & x \geq 0 \end{aligned}$	\Rightarrow	$\begin{aligned} z \quad & + \bar{c}_{B'}^T x_{B'} = \bar{v} \\ Ix_B + \bar{A}_{B'} x_{B'} & = \bar{b} \end{aligned}$
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a. Row operations to transform second row

$$A_B x_B + A_{B'} x_{B'} = b \quad \xrightarrow{\text{Multiply by } A_B^{-1}} \quad Ix_B + A_B^{-1} A_{B'} x_{B'} = A_B^{-1} b \quad (*)$$

Let $\bar{A}_{B'} = A_B^{-1} A_{B'}$

and $\bar{b} = A_B^{-1} b$

Constructing a Tableau

- Construct the tableau for basis B:

$\begin{array}{l} \max z \\ \text{s.t. } z - c^T x = 0 \\ Ax = b \\ x \geq 0 \end{array}$	\Rightarrow	$\begin{array}{l} z + \bar{c}_{B'}^T x_{B'} = \bar{v} \\ Ix_B + \bar{A}_{B'} x_{B'} = \bar{b} \end{array}$
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- Row operations to transform first row

$$z - c_B^T x_B - c_{B'}^T x_{B'} = 0$$

$$\oplus \quad c_B^T (x_B + A_B^{-1} A_{B'} x_{B'}) = c_B^T A_B^{-1} b \quad c_B^T (*)$$

$$z + c_B^T A_B^{-1} A_{B'} x_{B'} - c_{B'}^T x_{B'} = c_B^T A_B^{-1} b$$

$$\text{Let } \bar{c}_{B'}^T = c_B^T A_B^{-1} A_{B'} - c_{B'}^T \text{ and } \bar{v} = c_B^T A_B^{-1} b$$

Example: Tableau (1 of 3)

- $$z - x_1 - x_2 = 0 \quad (\text{a}) \quad x=(0,0,2,4)$$
- $$x_1 + x_3 = 2 \quad (\text{b}) \quad B=\{3,4\}$$
- $$x_1 + 2x_2 + x_4 = 4 \quad (\text{c})$$

- How obtain tableau for B={1,4}?
- Subtract (b) from (c); add (b) to (a):

- $$\begin{array}{rclcl} z & - & x_2 & + & x_3 & = & 2 \\ & & x_1 & & + & x_3 & = & 2 \\ & & 2x_2 & - & x_3 & + & x_4 & = & 2 \end{array} \quad \begin{array}{l} x=(2,0,0,2) \\ B=\{1,4\} \end{array}$$

- BFS? Objective?
- $x=(2,0,0,2)$; $z=2$. **An improvement!**

Example: Tableau (2 of 3)

$$\begin{array}{rclcl}
 \bullet & z & -x_2 & +x_3 & = 2 & x=(2,0,0,2) \\
 & & & & & B=\{1,4\} \\
 & x_1 & & +x_3 & = 2 & \\
 & & 2x_2 & -x_3 & +x_4 & = 2
 \end{array}$$

How can we improve further? *Have 2 enter basis. What leaves?*

- Increase $x_2:=t$; keep $x_3=0$; $z=2+t$
- Feasibility: $x_4=2-2t$.
- Set $x_2:=1$, because $x_4<0$ for $t>1$.
- New basis $B=\{1,2\}$. $x=(2,1,0,0)$, $z=3$.

Example: Tableau (3 of 3)

$$\begin{array}{rclcl}
 \bullet & z & -x_2 & +x_3 & = 2 & \text{(a)} & x=(2,0,0,2) \\
 & & & & & & B=\{1,4\} \\
 & x_1 & & +x_3 & = 2 & \text{(b)} & \\
 & & 2x_2 & -x_3 & +x_4 & = 2 & \text{(c)}
 \end{array}$$

Get tableau for $B=\{1,2\}$

$$\begin{array}{rclcl}
 \bullet & z & + & +\frac{1}{2}x_3 & +\frac{1}{2}x_4 & = 3 & x=(2,1,0,0) \\
 & & x_1 & + & x_3 & = 2 & B=\{1,2\} \\
 & & & x_2 & -\frac{1}{2}x_3 & +\frac{1}{2}x_4 & = 1
 \end{array}$$

- Obj coeff of nonbasic vars ≥ 0 . Solution optimal!
- Must have $z = 3 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \leq 3$ for any feasible solution, since $x_3, x_4 \geq 0$.

From Basis to Tableau and Back

- **Theorem.** If B is a basis for matrix A then there is a tableau for B . If there is a tableau for B then B is a basis for A .
- **Proof**
 - (\rightarrow) row operations to isolate the variables x_B (possible since A_B is invertible)
 - (\leftarrow) B is a basis because $A_B = I$, and therefore A_B spans.

Adjacent Bases

- **Definition.** Two bases for matrix A are **adjacent** if they share all but one column
- **Example**
 - $\max z = x_1 + x_2$
 - s.t.

$$\begin{array}{rcccc} x_1 & & & + x_3 & = 2 \\ x_1 & + 2x_2 & & + x_4 & = 4 \\ x_1, & x_2, & x_3, & x_4 & \geq 0 \end{array}$$
- Five bases $\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{3,4\}$.
- $B = \{1,2\}$ is adjacent to $\{1,3\}$, not $\{3,4\}$.

The Simplex Method

Work in tableau form.

- Step **(i)**. Find an initial BFS. ← **Next lecture**
- Step **(ii)**.
 - Determine if current BFS is optimal.
 - If not, find an adjacent BFS with a larger z-value.
- Repeat.

How fast is this?

- Number of possible BFS is “n choose m”
$${}^nC_m = n! / (n-m)! m!$$
- ${}^{20}C_{10} = 184,756$
- Even if we can avoid cycling... gets large.
 - “degeneracy” will lead to concern about cycling (next lecture)

Empirical performance

- Applied to LPs in canonical form, tends to find an optimal solution after searching less than **3m BFS**.
- Example: 20-variable, 10-constraint problem: search ≈ 30 BFS.
- Simplex has very good performance in practice

Simplex Method - Overview

- Canonical form:
$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & z - c^T x = 0 \\ & Ax = b, x \geq 0 \end{aligned}$$
- Step 0. Initialization
- Step 1. Check optimality.
- Step 2. Choose entering index.
- Step 3. Check unboundedness.
- Step 4. Choose leaving index.
- Step 5. Pivot to a new tableau. **To step 1.**

Simplex Method - Overview

- $\max z$
s.t. $z + \sum_{j \in B'} \bar{c}_j x_j = \bar{v}$
 $x_i + \sum_{j \in B'} \bar{a}_{ij} x_j = \bar{b}_i \quad (i \in B) \text{ (assume } \bar{b}_i \geq 0)$
- **Step 0.** Initialization (find a BFS and tableau)
- **Step 1.** If $\bar{c}_j \geq 0$ for all $j \in B'$, stop (optimal)
- **Step 2.** Else, pick **entering index** $k \in B'$ with $\bar{c}_k < 0$.
- **Step 3.** If $\bar{a}_{ik} \leq 0$ for all $i \in B$, stop (unbounded)
- **Step 4. Ratio test.** $t^* = \min\{\bar{b}_i / \bar{a}_{ik} : i \in B, \bar{a}_{ik} > 0\}$. Pick **leaving index** $r \in B$ with min ratio.
- **Step 5. Pivot** on (r,k) entry to get a tableau for $B := B \cup \{k\} \setminus \{r\}$.

Example of Simplex Method

		$z - x_1 - x_2$	$= 0$	Basic $x_3 \quad x_4$	
←	x_1	$+ x_3$	$= 2$	Ratio $2/1 \quad 4/1$	
	↑	$x_1 + 2x_2$	$+ x_4 = 4$		x_1 to enter. x_3 to leave. pivot(3,1)

		$z - x_2 + x_3$	$= 2$	Basic $x_1 \quad x_4$	
		$x_1 + x_3$	$= 2$	Ratio $2/2$	
←	$2x_2$	$- x_3 + x_4$	$= 2$		x_2 to enter. x_4 to leave. pivot(4,2)
	↑				

		$z + \frac{1}{2} x_3 + \frac{1}{2} x_4$	$= 3$	Basic $x_1 \quad x_2$	
		$x_1 + x_3$	$= 2$		
		$x_2 - \frac{1}{2} x_3 + \frac{1}{2} x_4$	$= 1$		Reduced costs all nonnegative. Stop!

- Solution: $(x_1, x_2, x_3, x_4) = (2, 1, 0, 0)$, $z = 3$.

Review

Step 0: Initialization

- Find a basis and initial tableau corresponding to a BFS
 - next lecture

Step 1: Check optimality

- If $\bar{c}_j \geq 0$ (for all j in B') then **stop, current solution is optimal**.
- Value $z = \bar{v}$. Any feasible solution satisfies

$$z = \bar{v} - \bar{c}^T x_{B'} \leq \bar{v},$$

since $\bar{c} \geq 0$ and $x \geq 0$.

- Non-negative reduced costs is a sufficient test for optimality. [It is a valid test even if a solution is “degenerate”, discussed in next lecture.]

Step 2: Choose Entering Index

- Pick some k in B' with $\bar{c}_k < 0$ to enter the basis, so that the obj value increases with x_k
- Say that \bar{c}_k is the “**reduced cost**” of nonbasic variable x_k . Amount by which z *decreases* when x_k increases (and so $\bar{c}_k < 0$ is good).

Step 3: Check Unboundedness

- $x_i + \sum_{j \in B'} \bar{a}_{ij} x_j = \bar{b}_i$ (for all $i \in B$)
- Because other nonbasic vars = 0, we can increase x_k while:
$$x_i = \bar{b}_i - \bar{a}_{ik} x_k \geq 0 \quad (\text{for all } i \in B)$$
- If $\bar{a}_{ik} \leq 0$ for every i in B , then x_k can increase without bound (since x_i can increase without affecting objective)!
- **Stop: objective value is unbounded.**

Step 4: Choose Leaving Index

- Need: $x_i = \bar{b}_i - \bar{a}_{ik} x_k \geq 0$; and so $x_k \leq \bar{b}_i / \bar{a}_{ik}$ (where $\bar{a}_{ik} > 0$).
- **Ratio test:** $t^* = \min\{ \bar{b}_i / \bar{a}_{ik} : i \text{ in } B, \bar{a}_{ik} > 0 \}$
- Let R denote basic vars with min ratio
- Pick some r in R as leaving index; set $x_k := t^*$ (may be zero)
- New basic variable x_k “kicks out” the “removed” variable x_r

Step 5: Pivot to new tableau

- **Definition.** A **pivot** on (r,k) is row operations to construct tableau for $B:=B\cup\{k\}\setminus\{r\}$.
- **(a)** Divide row $x_r + \sum_{j \in B} \bar{a}_{rj}x_j = \bar{b}_r$ through by \bar{a}_{rk} so that coefficient of new basic variable x_k becomes 1.
(Why does RHS remain nonnegative?)
- **(b)** Add/subtract multiples of this adjusted row to all other equations (including objective) to remove x_k
(Why do these operations not affect coeffs for other basic vars?)

[Note: we're doing "Gauss-Jordan elimination."]

Summary: Simplex Method

- A tableau is system of eqns that correspond to a basis and represent **all** details of the LP
– the basic variables are isolated
- The simplex method changes the tableau when searching through BFSs.
- Pivots through adjacent BFSs until prove LP is unbounded or find an optimal solution.
- Fast in practice!

Next lecture

- How to initialize the simplex method?
- Provable termination even when simplex method encounters a “degenerate BFS” (with a basic variable with value zero)?