AM 121: Intro to Optimization Models and Methods

Fall 2016

Lecture 13: Branch and Bound (I)

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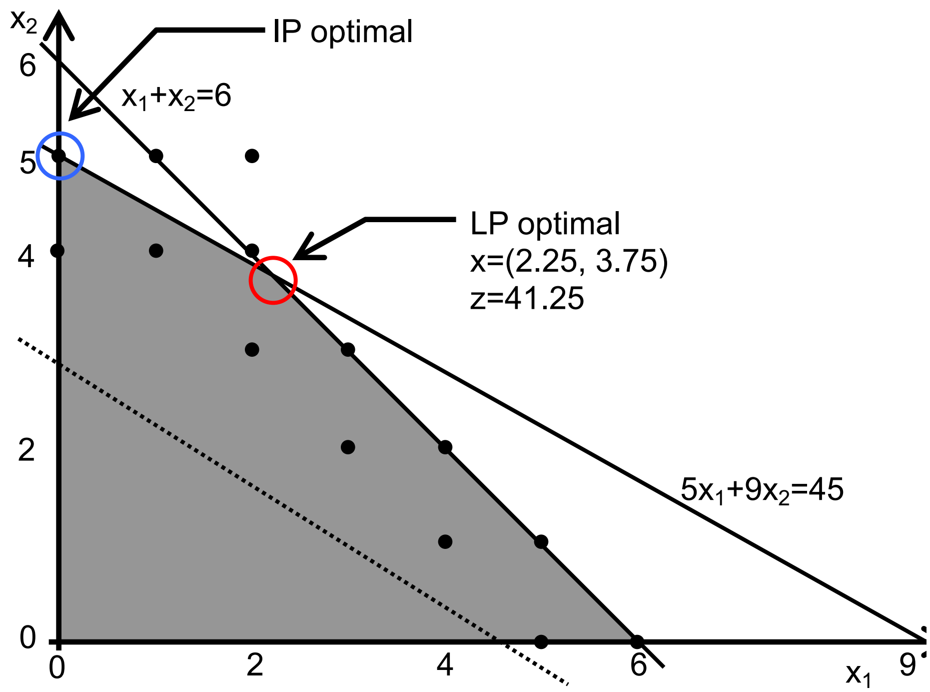
SEAS

- **Example**:

$$\max 5x_1 + 8x_2$$
$$\text{s.t. } x_1 + x_2 \leq 6$$
$$5x_1 + 9x_2 \leq 45$$
$$x_1, \quad x_2 \geq 0, \text{ integer}$$



Lesson Plan: Solving IPs

- LP relaxations
- The “branch and bound” method
- Correctness of BnB

Jensen & Bard: 8.3

LP Relaxations

- A **polyhedron** is a set that can be described in the form $P = \{x \in R^n : Ax \leq b\}$
- The feasible set of any LP can be described as a polyhedron.

- **Definition**. The **LP relaxation** of an integer program replaces all integer variables with continuous variables.

- **Definition**. The **polyhedron (**P**) of an IP** is the feasible region for the LP relaxation.

LP Relaxations

- (IP) $z = \max \{c^T x: Ax \leq b, x \geq 0, x \text{ integer}\}$
- Write this as
- $z = \max \{c^T x: x \in P, x \text{ integer}\}$
- Let $S = P \cap Z^n$ (Z = *integers*) denote the feasible solution set of the IP
- **Defn.** The polyhedron of an IP is:
  $P=\{x \in R^n: Ax \leq b, x \geq 0\}$.

- **Linear programming relaxation** (LPR) of IP is $z_{LP} = \max\{c^T x: x \in P\}$

Example 1 (LPR)

max $7x_1+4x_2+5x_3+2x_4$

s.t. $3x_1+3x_2+4x_3+2x_4 \leq 6$

$x \in \{0,1\}^4$

Let S denote feasible solution space of IP

- $P=\{x \in R^4: 3x_1+3x_2+4x_3+ 2x_4 \leq 6, x \geq 0, x \leq 1\}$
- Solution to LPR is $x_{LP}=(1,1,0,0)$
- Integral ➔ also a solution to IP.

Example 2 (LPR)

$z=\max 4x_1 - x_2$

s.t. $7x_1-2x_2 \leq 14$

$x_2 \leq 3$

$2x_1-2x_2 \leq 3$

$x_1, \quad x_2 \geq 0$, integer

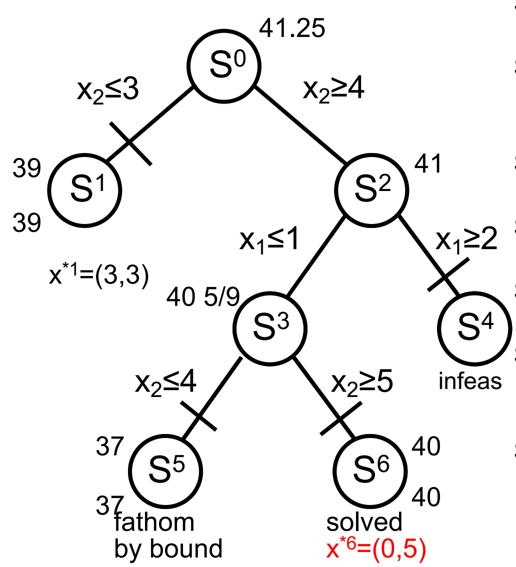
Let S denote feasible solution space of IP

- Feasible solution: x=(2,1), so $7 \leq z$.
- Solve LPR: $x_{LP}=(20/7, 3)$, $z_{LP}=59/7 =8.42$
- Conclude: $z \leq 8.42$
- Moreover: objective coeffs in S are integers, x_1, x_2 are integer, and so:

$$z \leq \lfloor z_{\mathrm{LP}} \rfloor = 8$$

Example of Branch and Bound

$$\begin{aligned} \max\ & 5x_1 + 8x_2 \\ \text{s.t. } & x_1 + \quad x_2 \leq 6 \\ & 5x_1 + 9x_2 \leq 45 \\ & x_1, \quad x_2 \geq 0, \text{ integer} \end{aligned}$$



best soln → upper bound

S^0: (2.25, 3.75) $-\infty$, 41.25
fractional, branch on x_2

S^1: (3,3) 39, 41.25
fathom by integrality
new solution!

S^2: (1.8, 4) 39, 41
fractional, branch on x_1

S^3: (1, 4 4/9) 39, 41
fractional, branch on x_2

S^5: (1,4) 39, 41
fathom by bound

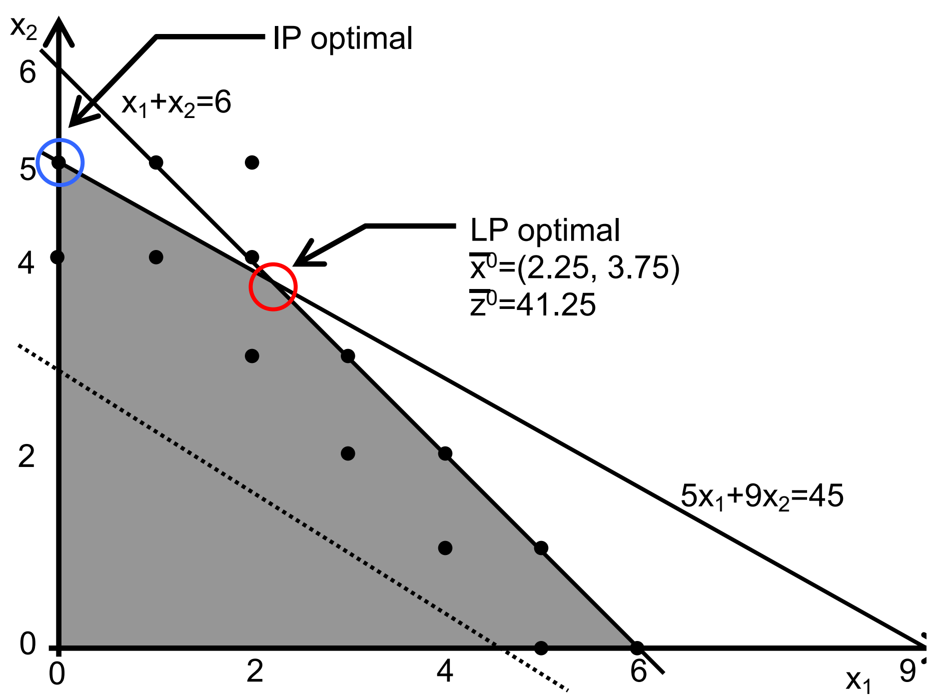
S^6: (0,5) 40, 41
fathom by integrality
new solution!

S^4: infeasible 40, 40
fathom by infeasibility

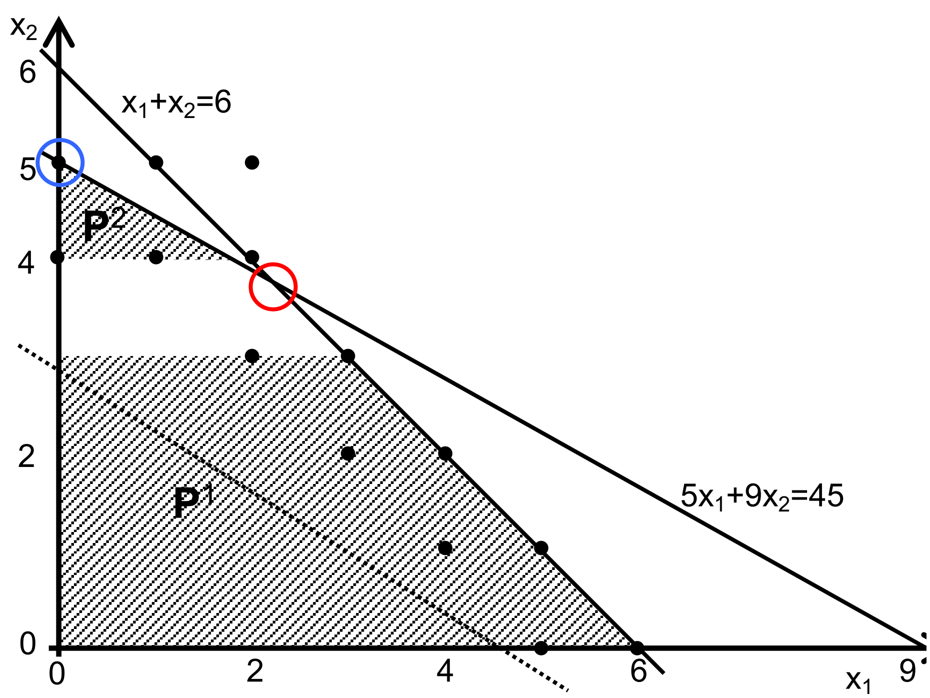
Branch and Bound

- Solve LP relaxation and branch if solution is not integral. Generate subproblems.
- If value of LPR of subproblems is larger than best integer solution so far then can avoid solving subproblem ("prune by bound.")
- **Example**:

$$\max\ 5x_1 + 8x_2$$
$$\text{s.t. } x_1 + x_2 \leq 6$$
$$5x_1 + 9x_2 \leq 45$$
$$x_1, \quad x_2 \geq 0, \text{ integer}$$



- Let S^0 denote feasible solution space of IP
- Consider *LP relaxation*.
- Let $P^0=\{x \in R^2: x_1+x_2 \leq 6, 5x_1+9x_2 \leq 45, x \geq 0\}$. $z^0=41.25$, $x^0=(2.25,3.75)$
- Variable x_2 should be integer, and so valid to consider subproblems with $x_2 \leq 3$ and $x_2 \geq 4$.
- **Branch**!
- Let S^1 denote $S^0 \cap \{x_2 \leq 3\}$.
- Let S^2 denote $S^0 \cap \{x_2 \geq 4\}$.



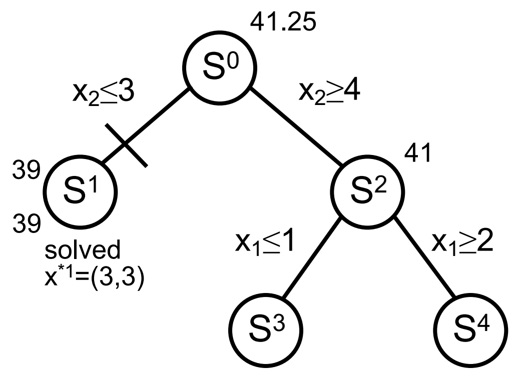
- Two new IPs, with solution spaces S^1 and S^2. We have $S^1 \cup S^2 = S$ and $S^1 \cap S^2 = \emptyset$
- Call S^1 and S^2 **subproblems,** and this process **branching**.

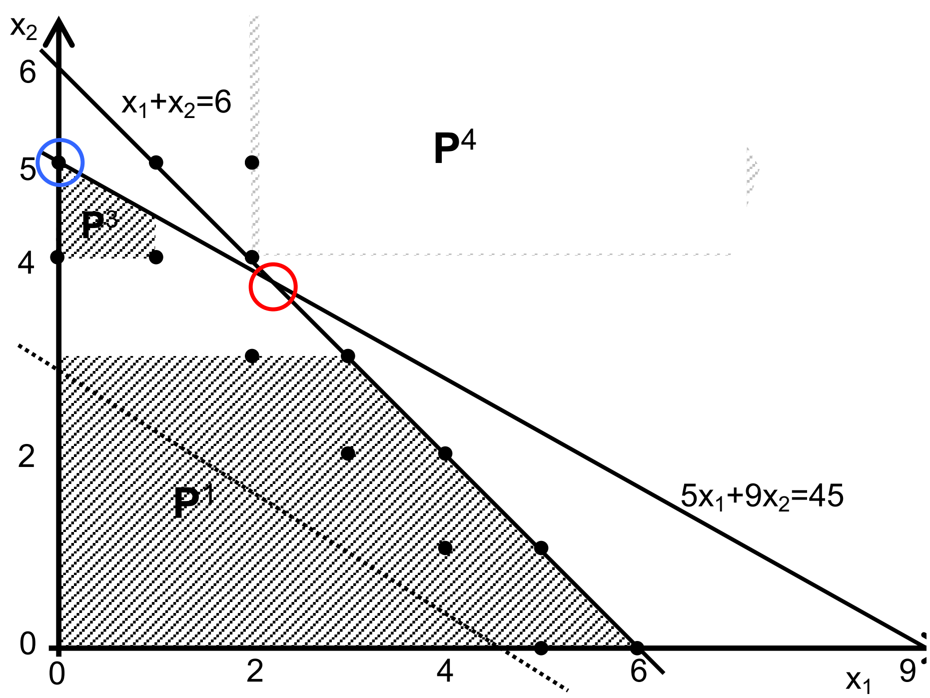
- Consider S^1 first. Optimal solution to LP(S^1) is $\bar{x}^1=(3,3)$, $\bar{z}^1=39$. Our first feasible solution; write $x^{*1}=(3,3)$, $z^{*1}=39$.

- Maintain an **incumbent**, $\underline{z}:=\max(\underline{z}, z^{*1}) = \max(-\infty, 39) = 39$, and $\underline{x}:=x^{*1}=(3,3)$.

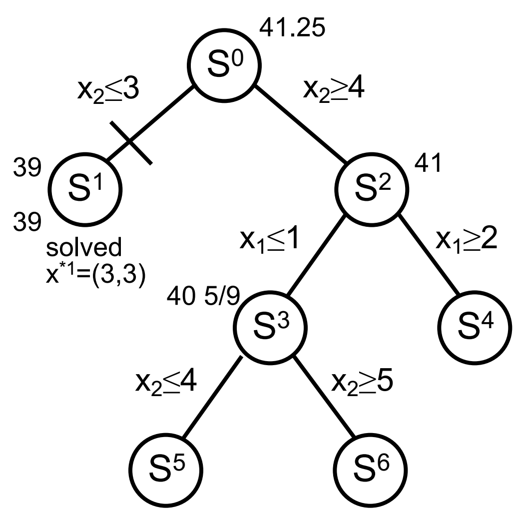
- Now consider S^2.
- Optimal solution to LP(S^2) is fractional.
- $\bar{x}^2=(1.8, 4)$, $\bar{z}^2=41$
- $\bar{z}^2 > \underline{z} = 39$ and so S^2 could include a better IP solution than incumbent.
- **Branch** again!
- $S^3 = S^0 \cap \{x_2 \geq 4, x_1 \leq 1\}$
- $S^4 = S^0 \cap \{x_2 \geq 4, x_1 \geq 2\}$
- $S=S^1 \cup S^3 \cup S^4$, $S^1 \cap S^3 \cap S^4 = \emptyset$

- Develop a **search tree** of subproblems

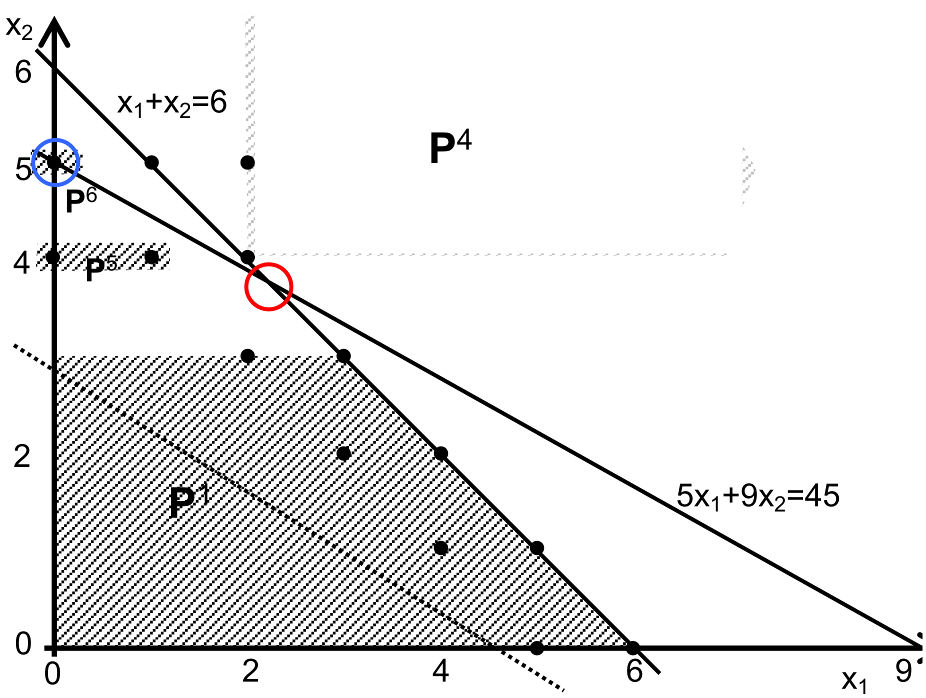


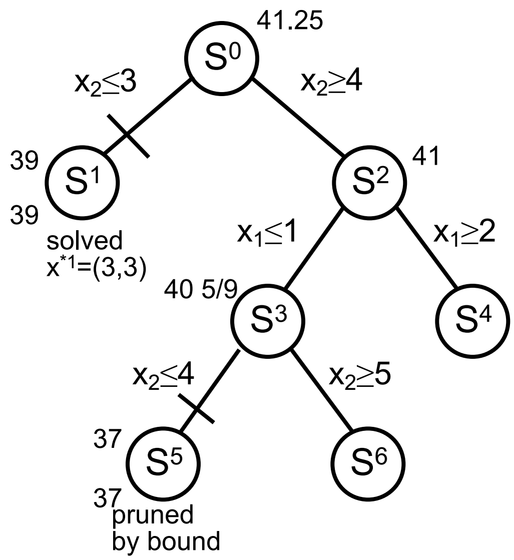


- Consider left branch. Solve LP(S^3)
- $\bar{x}^3$=(1, 4 4/9), $\bar{z}^3$ = 40 5/9
- $\bar{z}_3 > \underline{z} = 39$. Branch again!
- $S^5 = S^0 \bigcap \{x_2 \geq 4, x_1 \leq 1, x_2 \leq 4\} = S^0 \bigcap \{x_1 \leq 1, x_2 = 4\}$
- $S^6 = S^0 \bigcap \{x_2 \geq 4, x_1 \leq 1, x_2 \geq 5\} = S^0 \bigcap \{x_1 \leq 1, x_2 \geq 5\}$



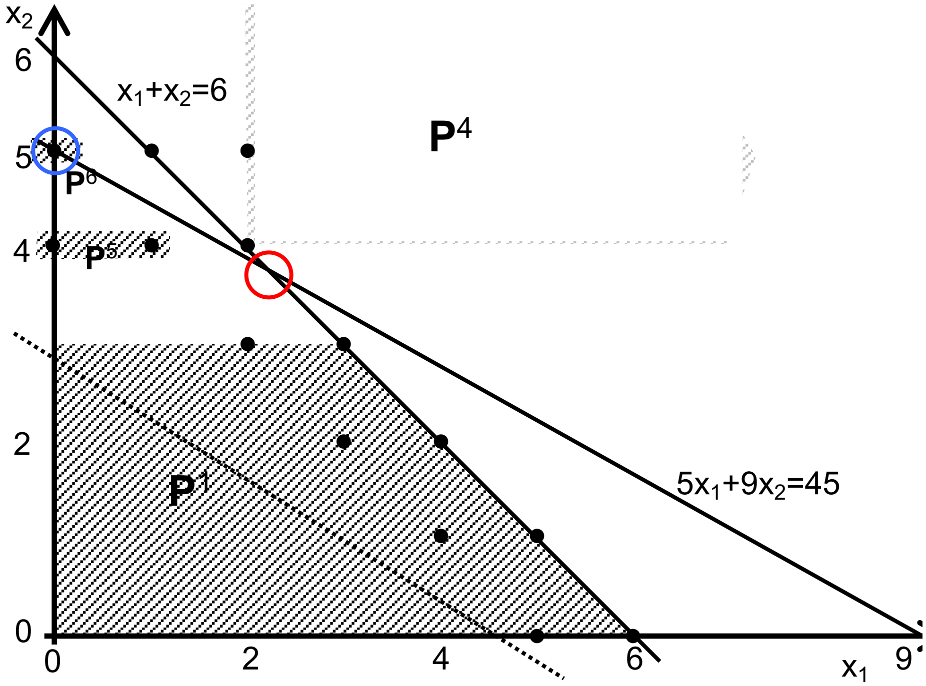
- Have 3 possible directions.
- Could consider subproblems S^4, S^5 or S^6
- Suppose **depth first search:** consider a child of the most recently considered subproblem. Take the **left branch**. Solve LP(S^5).
- $\overline{x}^5$= (1,4), $\overline{z}^5$ = 37
- $\overline{z}^5 \leq \underline{z}$ = 39. Can **prune** S^5; best possible soln. is bounded above by 37.
- This node is **fathomed by bound**.
- “Branch and bound”!



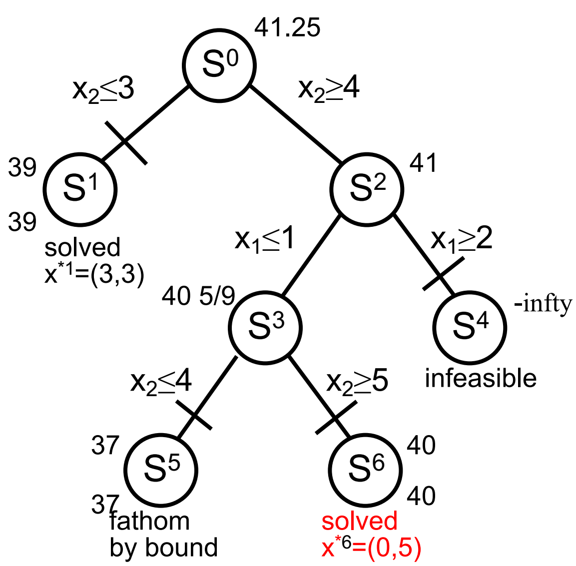


- Now at a leaf. Because using DFS order, we back-up and find the first node with an unsolved problem.
- Right branch under S^3 (i.e., S^6).
- Solve LP(S^6)
- $\bar{z}^6$=40, $\bar{x}^6$=(0,5). Another feasible solution!
- Have: z^{*6}=40, x^{*6}=(0,5)
- $\underline{z}$:=max($\underline{z}$,z^*_6) =max(39,40)=40, $\underline{x}$:=(0,5).
- New incumbent solution.

- Last step: Solve S^4. LP(S^4) is infeasible.



Final search tree. Completely “fathomed” (or “solved”)



- No branches left to explore.
- Conclude that $x^*=\underline{x}=(0,5)$, $z^*=\underline{z}=40$ is the optimal solution.

Review

$$
\begin{aligned}
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\text{s.t. } & x_1 + x_2 \leq 6 \\
& 5x_1 + 9x_2 \leq 45 \\
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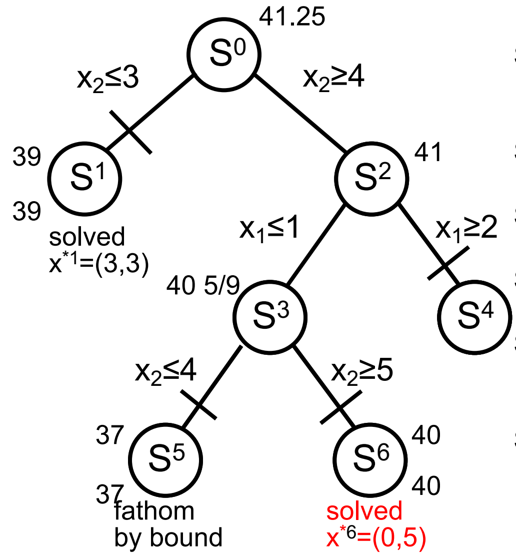
$$



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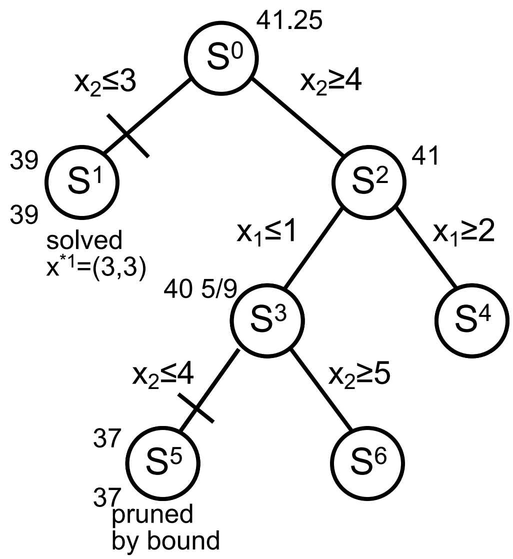
BnB: Summary

- Maintain a list of **open subproblems**, an **incumbent** $\underline{x}$ with value $\underline{z}$, and an upper bound $\bar{z}$ (maximal over bounds of parents of open nodes)
- At each step:
  - Node selection decision: pick an open subproblem and solve the LP relaxation
  - Fathom the node (by infeasibility, by bound, or by integrality -- update the incumbent if better).
  - Branching decision: if can't fathom node, pick a fractional variable in the LPR solution and branch
- Continue until all open subproblems are fathomed, or the **optimality gap** is acceptable.

Optimality gap



Upper bound
= max(40 5/9, 41) = 41
Best solution = 39
Absolute Gap = 41 – 39 = 2
Relative Gap = (2 / 39)*100%
= 5.1%

Correctness of BnB

- Consider (IP) max $\{c^T x: x \in S\}$
- Divide $S= S^1 \cup S^2 \ldots \cup S^K$, represented by the leaves of the final tree
- **Proposition.** Given a fathomed search tree, the final incumbent (z^*,x^*) solves $z^*=\max_k z^{*k}$ where $z^{*k} = \max\{c^T x: x \in S^k\}$.
- **Proof:** For each S^k, problem IP^k is either
  - (i) **infeasible, or**
  - (ii) **pruned by bound**, value $LP(S^k) \le \underline{z} \le z^*$, **or**
  - (iii) **solved.**

Properties of Relaxations

$$A: \ z_A = \max\{\boldsymbol{c}_A^\top \boldsymbol{x} : \boldsymbol{x} \in S_A\}$$

$$B: \ z_B = \max\{\boldsymbol{c}_B^\top \boldsymbol{x} : \boldsymbol{x} \in S_B\}$$

Definition 1. *A is a relaxation of B if:*
(1) $S_B \subseteq S_A$
(2) $\boldsymbol{c}_B^\top \boldsymbol{x} \leq \boldsymbol{c}_A^\top \boldsymbol{x}$*, for all* $x \in S_B$

Theorem 1. *If A is a relaxation of B, then* $z_A \geq z_B$.

Proof. Let $\boldsymbol{x}_A, \boldsymbol{x}_B$ denote optimal solutions to A and B, respectively.

$$z_A = \boldsymbol{c}_A^\top \boldsymbol{x}_A \geq \boldsymbol{c}_A^\top \boldsymbol{x}_B \geq \boldsymbol{c}_B^\top \boldsymbol{x}_B = z_B$$

where first inequality is by optimality of $\boldsymbol{x}_A$ and (1) of relaxation, and second inequality is by (2) of relaxation. □

Properties of Relaxations

$$A: \ z_A = \max\{\boldsymbol{c}_A^\top \boldsymbol{x} : \boldsymbol{x} \in S_A\}$$

$$B: \ z_B = \max\{\boldsymbol{c}_B^\top \boldsymbol{x} : \boldsymbol{x} \in S_B\}$$

Definition 1. *A is a relaxation of B if:*
(1) $S_B \subseteq S_A$
(2) $\boldsymbol{c}_B^\top \boldsymbol{x} \leq \boldsymbol{c}_A^\top \boldsymbol{x}$*, for all* $x \in S_B$

Theorem 2. *If A is infeasible then B is infeasible.*

Proof. Because $\emptyset = S_A \supseteq S_B$.

Properties of Relaxations

$$A: \; z_A = \max\{\boldsymbol{c}_A^\top \boldsymbol{x} : \boldsymbol{x} \in S_A\}$$
$$B: \; z_B = \max\{\boldsymbol{c}_B^\top \boldsymbol{x} : \boldsymbol{x} \in S_B\}$$

Definition 1. *A is a relaxation of B if:*
(1) $S_B \subseteq S_A$
(2) $\boldsymbol{c}_B^\top \boldsymbol{x} \leq \boldsymbol{c}_A^\top \boldsymbol{x}$, *for all* $x \in S_B$

Theorem 3. *Suppose* $\boldsymbol{c}_A = \boldsymbol{c}_B = \boldsymbol{c}$. *Then, if* $\boldsymbol{x}_A \in S_B$ *then* $\boldsymbol{x}_A$ *solves* B.

Proof. Suppose $\boldsymbol{x} \in S_B$ has $\boldsymbol{c}^\top \boldsymbol{x} > \boldsymbol{c}^\top \boldsymbol{x}_A$. Contradiction with optimality of $\boldsymbol{x}_A$ in A since $\boldsymbol{x} \in S_A$. □

Properties of LP Relaxations

Definition 1. *A is a relaxation of B if:*
(1) $S_B \subseteq S_A$
(2) $\boldsymbol{c}_B^\top \boldsymbol{x} \leq \boldsymbol{c}_A^\top \boldsymbol{x}$, *for all* $x \in S_B$

- LPR is a relaxation of an IP
  - Feas. solution set larger, objective same.
- We have:
  1. $z_{LP} \geq z_{IP}$ (if maximization)
  2. If LPR infeasible then IP is infeasible
  3. If LPR solution is integral, then this solution also solves the IP.

Solving Mixed Integer Programs

- Both integer and continuous variables.
- Branch and Bound immediately extends.
- Only branch on variables that are required to be integral in feasible solution.

Summary

- Branch and Bound is a general purpose methodology for solving IPs
- Crucial ingredients:
  - Branching on a variable with a fractional assignment to create subproblems and make progress
  - Using the LP relaxation to bound away whole parts of the search space
  - Finding integral solutions to LP relaxations