AM 121: Intro to Optimization Models and Methods

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Lecture 10: Dual Simplex

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Lesson Plan

- Interpret primal simplex in terms of pivots on the corresponding **dual tableau**
- **Dictionaries**
- Define **dual simplex** and interpret it in terms of pivots on the primal tableau
- **Applications**

Jensen & Bard: 3.9

Motivation: Dual Simplex

- Useful for solving **integer programs** via branch-and-bound search.
- Find a solution after a new constraint is added, or an existing constraint is modified.
- A new "Phase 1 – Phase 2" method (no need for artificial variables!)

Naming convention

- max $\sum_{j=m+1}^{m+n} c_j x_j$
  s.t. $\sum_{j=m+1}^{m+n} a_{ij}x_j \leq b_i$ $\quad i = 1, 2, \ldots, m$
  $x_j \geq 0$ $\quad j = m+1, \ldots, m+n$

  primal vars $x_{m+1},\ldots,x_{m+n}$; slack vars $x_1,\ldots,x_m$

- min $\sum_{i=1}^{m} b_i y_i$
  s.t. $\sum_{i=1}^{m} a_{ij}y_i \geq c_j$ $\quad j = m+1, \ldots, m+n$
  $y_i \geq 0$ $\quad i = 1, \ldots, m$

  dual vars $y_1,\ldots,y_m$; excess vars $y_{m+1},\ldots,y_{m+n}$

Example

max $4x_3 + x_4 + 3x_5$
s.t. $x_3 + 4x_4 \leq 1$
$3x_3 - x_4 + x_5 \leq 3$
$x_3, x_4, x_5 \geq 0$

Dual:

- min $y_1 + 3y_2$
  s.t. $y_1 + 3y_2 \geq 4$
  $4y_1 - y_2 \geq 1$
  $y_2 \geq 3$
  $y_1, y_2 \geq 0$

$\longrightarrow$ equivalent maximization problem

max $-y_1 - 3y_2$
s.t. $-y_1 - 3y_2 \leq -4$
$-4y_1 + y_2 \leq -1$
$-y_2 \leq -3$
$y_1, y_2 \geq 0$

Tableau (Review)

- A tableau must have isolated variables but **does not need to have** non-negative right hand side values.
- If the RHS is negative, the tableau is not (primal) feasible.

Primal <> Dual

Initial **primal tableau**:

$$\begin{array}{lllllll} z & & & -4x_3 & -x_4 & -3x_5 & = 0 \\ & x_1 & & +x_3 & +4x_4 & & = 1 \\ & & x_2 & +3x_3 & -x_4 & +x_5 & = 3 \end{array}$$

- $B=\{1,2\}$ $B'=\{3,4,5\}$

feasible

Initial **dual tableau**:

$$\begin{array}{llllll} z & +y_1 & +3y_2 & & & = 0 \\ & -y_1 & -3y_2 & +y_3 & & = -4 \\ & -4y_1 & +y_2 & & +y_4 & = -1 \\ & & -y_2 & & +y_5 & = -3 \end{array}$$

- $B=\{3,4,5\}$ $B'=\{1,2\}$

infeasible

RHS

non-basic

$$\begin{array}{c|cccc} & & x_3 & x_4 & x_5 \\ \hline z & 0 & 4 & 1 & 3 \\ x_1 & 1 & -1 & -4 & 0 \\ x_2 & 3 & -3 & 1 & -1 \end{array}$$

(negated columns)

basic

Primal dictionary

non-basic

$$\begin{array}{c|ccc} & & y_1 & y_2 \\ \hline z & 0 & -1 & -3 \\ y_3 & -4 & 1 & 3 \\ y_4 & -1 & 4 & -1 \\ y_5 & -3 & 0 & 1 \end{array}$$

(negated columns)

basic

Dual dictionary

RHS

Primal <> Dual

Initial **primal tableau**:

$$\begin{array}{lllllll} z & & & -4x_3 & -x_4 & -3x_5 & = 0 \\ & x_1 & & +x_3 & +4x_4 & & = 1 \\ & & x_2 & +3x_3 & -x_4 & +x_5 & = 3 \end{array}$$

- $B=\{1,2\}$ $B'=\{3,4,5\}$

Feasible; Neg shadow costs

Initial **dual tableau**:

$$\begin{array}{llllll} z & +y_1 & +3y_2 & & & = 0 \\ & -y_1 & -3y_2 & +y_3 & & = -4 \\ & -4y_1 & +y_2 & & +y_4 & = -1 \\ & & -y_2 & & +y_5 & = -3 \end{array}$$

- $B=\{3,4,5\}$ $B'=\{1,2\}$

Infeasible; Non-neg shadow costs

$$\begin{pmatrix} 0 & 4 & 1 & 3 \\ 1 & -1 & -4 & 0 \\ 3 & -3 & 1 & -1 \end{pmatrix} \quad B=\{1,2\}$$

Primal dictionary

$$\begin{pmatrix} 0 & -1 & -3 \\ -4 & 1 & 3 \\ -1 & 4 & -1 \\ -3 & 0 & 1 \end{pmatrix} \quad B=\{3,4,5\}$$

Dual dictionary

-ve transpose of primal;
B (dual) = B' (primal)

A Dictionary

- **Definition.** The dictionary corresponding to a tableau with basis B is $\begin{pmatrix} v & -\bar{c}_{B'}{}^{T} \\ \bar{b} & -\bar{A}_{B'} \end{pmatrix}$ (where B' is the set of non-basic)

  with rows of $\bar{A}_{B'}$ and $\bar{b}$ **ordered by indices** of basic vars and columns of $\bar{c}_{B'}{}^{T}$ **ordered by indices** of non-basic vars
- Suppose tableau was reordered as:

$$\begin{array}{lllll} z & & - x_4 - 4x_3 - 3x_5 & = 0 \\ & x_2 & - x_4 + 3x_3 + x_5 & = 3 \\ x_1 & & + 4x_4 + x_3 & = 1 \end{array}$$

  Corresponding dictionary is still:

$$\begin{pmatrix} 0 & 4 & 1 & 3 \\ 1 & -1 & -4 & 0 \\ 3 & -3 & 1 & -1 \end{pmatrix} \quad B=\{1,2\}$$

- **A dictionary and basis B completely defines a tableau**!

A simple primal-dual correspondence

$$\text{Primal dictionary} \xleftrightarrow{\text{negated transpose}} \text{Dual dictionary}$$

$$\begin{array}{ccc} 0 & 4 & 1 & 3 \\ 1 & -1 & -4 & 0 \\ 3 & -3 & 1 & -1 \end{array} \qquad\qquad \begin{array}{ccc} 0 & -1 & -3 \\ -4 & 1 & 3 \\ -1 & 4 & -1 \\ -3 & 0 & 1 \end{array}$$

Basic B, non-basic B' Basic B', non-basic B

So: (1) primal tableau has corresponding dual tableau
(2) can track progress of simplex in the dual!
(3) primal obj value = - dual obj value at each step
(negated because we converted min f(y) into max –f(y))

Primal pivot (track in dual)

$$\begin{aligned} z \quad - 4x_3 - x_4 - 3x_5 &= 0 \\ x_1 + x_3 + 4x_4 &= 1 \\ x_2 + 3x_3 - x_4 \boxed{+ x_5} &= 3 \end{aligned}$$

$B=\{1,2\}$ $B'=\{3,4,5\}$

pivot (2,5); 2 out 5 in

$$\xrightarrow[\text{pivot}]{}$$

$$\begin{aligned} z \quad +3x_2+5x_3 - 4x_4 &= 9 \\ x_1 +x_3 + 4x_4 &= 1 \\ x_2 +3x_3 - x_4 +x_5 &= 3 \end{aligned}$$

$B=\{1,5\}$ $B'=\{2,3,4\}$

$$\begin{aligned} z +y_1 +3y_2 &= 0 \\ -y_1 -3y_2 +y_3 &= -4 \\ -4y_1 + y_2 +y_4 &= -1 \\ \boxed{- y_2} +y_5 &= -3 \end{aligned}$$

$B=\{3,4,5\}$ $B'=\{1,2\}$

pivot (5,2); 5 out 2 in

$$\xrightarrow[\text{pivot}]{}$$

$$\begin{aligned} z +y_1 +3y_5 &= -9 \\ -y_1 +y_3 -3y_5 &= 5 \\ -4y_1 +y_4+y_5 &= \boxed{-4} \\ y_2 -y_5 &= 3 \end{aligned}$$

$B=\{2,3,4\}$ $B'=\{1,5\}$

still dual infeasible

New dictionaries

$$\begin{aligned} z +3x_2 +5x_3-4x_4 &= 9 \\ x_1 +x_3 +4x_4 &= 1 \\ x_2 +3x_3 -x_4 +x_5 &= 3 \end{aligned}$$

$B=\{1,5\}$ $B'=\{2,3,4\}$

$$\begin{array}{c|cccc} & & x_2 & x_3 & x_4 \\ \hline z & 9 & -3 & -5 & 4 \\ x_1 & 1 & 0 & -1 & -4 \\ x_5 & 3 & -1 & -3 & 1 \end{array}$$

Primal dictionary

$$\begin{aligned} z +y_1 +3y_5 &= -9 \\ -y_1 +y_3 -3y_5 &= 5 \\ -4y_1 +y_4+y_5 &= -4 \\ y_2 -y_5 &= 3 \end{aligned}$$

$B=\{2,3,4\}$ $B'=\{1,5\}$

$$\begin{array}{c|ccc} & & y_1 & y_5 \\ \hline z & -9 & -1 & -3 \\ y_2 & 3 & 0 & 1 \\ y_3 & 5 & 1 & 3 \\ y_4 & -4 & 4 & -1 \end{array}$$

Dual dictionary

(**negated transpose**)

Primal pivot (track in dual)

$$\begin{array}{l} z +3x_2+5x_3 -4x_4 = 9 \\ x_1 \quad +x_3 \boxed{+4x_4} = 1 \\ \quad x_2 +3x_3 -x_4 +x_5 = 3 \\ B=\{1,5\}\ B'=\{2,3,4\} \end{array} \xrightarrow{\text{pivot}} \begin{array}{l} z +x_1+3x_2 \quad +6x_3 = 10 \\ \frac{1}{4}x_1 \quad +\frac{1}{4}x_3 + x_4 = \frac{1}{4} \\ \frac{1}{4}x_1 +x_2 +3\frac{1}{4}x_3 + x_5 = 3\frac{1}{4} \\ B=\{4,5\}\ B'=\{1,2,3\} \end{array}$$

Pivot (1,4); 1 out 4 in

Primal feasible,
Primal optimal.

$$\begin{array}{l} z +y_1 \quad +3y_5 = -9 \\ -y_1 \quad +y_3 \quad -3y_5 = 5 \\ \boxed{-4y_1} \quad +y_4+y_5 = -4 \\ \quad y_2 \quad -y_5 = 3 \\ B=\{2,3,4\}\ B'=\{1,5\} \end{array} \xrightarrow{\text{pivot}} \begin{array}{l} z \quad +\frac{1}{4}y_4 + 3\frac{1}{4}y_5 = -10 \\ y_1 \quad - \frac{1}{4}y_4 - \frac{1}{4}y_5 = 1 \\ \quad y_2 \quad -y_5 = 3 \\ \quad y_3 -\frac{1}{4}y_4 -3\frac{1}{4}y_5 = 6 \\ B=\{1,2,3\}\ B'=\{4,5\} \end{array}$$

Pivot (4,1); 4 out 1 in

Dual feasible,
Dual optimal

Final tableaus; Final dictionaries

$$\begin{array}{l} z +x_1+3x_2 \quad +6x_3 = 10 \\ \frac{1}{4}x_1 \quad +\frac{1}{4}x_3 + x_4 = \frac{1}{4} \\ \frac{1}{4}x_1 +x_2 +3\frac{1}{4}x_3 + x_5 = 3\frac{1}{4} \\ B=\{4,5\}\ B'=\{1,2,3\} \end{array}$$

$$\begin{array}{c|cccc} & & x_1 & x_2 & x_3 \\ \hline z & 10 & -1 & -3 & -6 \\ x_4 & \frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} \\ x_5 & 3\frac{1}{4} & -\frac{1}{4} & -1 & -3\frac{1}{4} \end{array}$$

Primal dictionary

$$\begin{array}{l} z \quad +\frac{1}{4}y_4 + 3\frac{1}{4}y_5 = -10 \\ y_1 \quad - \frac{1}{4}y_4 - \frac{1}{4}y_5 = 1 \\ \quad y_2 \quad -y_5 = 3 \\ \quad y_3 -\frac{1}{4}y_4 -3\frac{1}{4}y_5 = 6 \\ B=\{1,2,3\}\ B'=\{4,5\} \end{array}$$

$$\begin{array}{c|ccc} & & y_4 & y_5 \\ \hline z & -10 & -\frac{1}{4} & -3\frac{1}{4} \\ y_1 & 1 & \frac{1}{4} & \frac{1}{4} \\ y_2 & 3 & 0 & 1 \\ y_3 & 6 & \frac{1}{4} & 3\frac{1}{4} \end{array}$$

Dual dictionary
(**negated transpose**)

From Initial to Final tableau

Initial primal tableau

$$\begin{aligned} z \quad &- 4x_3 - x_4 - 3x_5 = 0 \\ x_1 \quad &+ x_3 + 4x_4 = 1 \\ x_2 &+ 3x_3 - x_4 + x_5 = 3 \end{aligned}$$

B={1,2} B’={3,4,5}

Final primal tableau

$$\begin{aligned} z + x_1 + 3x_2 \quad + 6x_3 &= 10 \\ \tfrac{1}{4}x_1 \quad + \tfrac{1}{4}x_3 + x_4 &= \tfrac{1}{4} \\ \tfrac{1}{4}x_1 + x_2 + 3\tfrac{1}{4}x_3 + x_5 &= 3\tfrac{1}{4} \end{aligned}$$

B={4,5} B’={1,2,3}

Initial dual tableau

$$\begin{aligned} z + y_1 + 3y_2 &= 0 \\ -y_1 - 3y_2 + y_3 &= -4 \\ -4y_1 + y_2 + y_4 &= -1 \\ -y_2 + y_5 &= -3 \end{aligned}$$

B={3,4,5} B’={1,2}

Final dual tableau

$$\begin{aligned} z \quad + \tfrac{1}{4}y_4 + 3\tfrac{1}{4}y_5 &= -10 \\ y_1 \quad - \tfrac{1}{4}y_4 - \tfrac{1}{4}y_5 &= 1 \\ y_2 \quad - y_5 &= 3 \\ y_3 - \tfrac{1}{4}y_4 - 3\tfrac{1}{4}y_5 &= 6 \end{aligned}$$

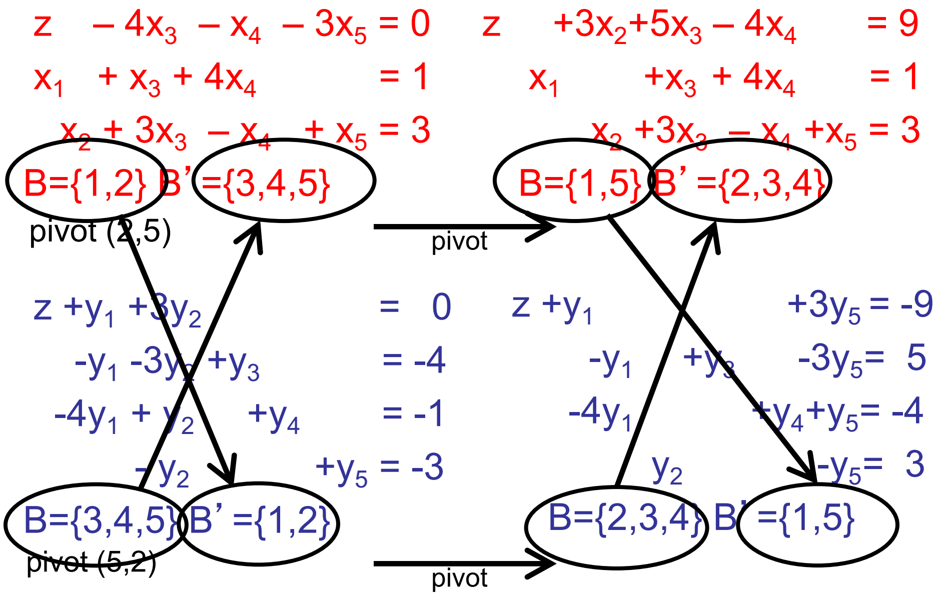
B={1,2,3} B’={4,5}

Primal Simplex: Effect On Dual Solution

- Maintains primal feasibility, terminates with primal optimality
  - Non-negative RHS in primal <> non-negative reduced costs in dual
  - Free reduced costs in primal <> free RHS values in dual (and dual infeasible for a while)
  - Terminates with non-negative reduced costs in primal <> non-neg RHS in dual (and dual feasible)
  - “Pivot in primal, track in dual.”
- Corresponding dual solution always has non-negative reduced costs, but is initially infeasible.

Note: pair (x,y) satisfy complementary slackness during pivots, but y is infeasible until termination.

Complementary Slackness



$z \quad - 4x_3 \; - x_4 \; - 3x_5 = 0$

$x_1 \quad + x_3 + 4x_4 \qquad = 1$

$x_2 + 3x_3 \; - x_4 \; + x_5 = 3$

B={1,2} B'={3,4,5}

pivot (2,5)

$z \quad +3x_2+5x_3 - 4x_4 \qquad = 9$

$x_1 \qquad +x_3 + 4x_4 \qquad = 1$

$x_2 +3x_3 \; - x_4 +x_5 = 3$

B={1,5} B'={2,3,4}

$z +y_1 +3y_2 \qquad = 0$

$-y_1 -3y_2 +y_3 \qquad = -4$

$-4y_1 + y_2 \qquad +y_4 \qquad = -1$

$-y_2 \qquad +y_5 = -3$

B={3,4,5} B'={1,2}

pivot (5,2)

$z +y_1 \qquad +3y_5 = -9$

$-y_1 \quad +y_3 \qquad -3y_5 = 5$

$-4y_1 \qquad +y_4+y_5 = -4$

$y_2 \qquad -y_5 = 3$

B={2,3,4} B'={1,5}

It is because of CS that we get this particular dual <> primal construction

Primal dictionary ←→ Dual dictionary

$$\begin{array}{rrrr} 0 & 4 & 1 & 3 \\ 1 & -1 & -4 & 0 \\ 3 & -3 & 1 & -1 \end{array} \qquad \text{negated transpose} \qquad \begin{array}{rrr} 0 & -1 & -3 \\ -4 & 1 & 3 \\ -1 & 4 & -1 \\ -3 & 0 & 1 \end{array}$$

Basic B, non-basic B'

Basic B', non-basic B

In particular, it is CS that gives us that:

primal obj value = - dual obj value at each step
(negated because we converted min f(y) into max –f(y))

Different kinds of pivots

- **V1**: pick non-basic x_j with $\bar{c}_j<0$ to enter, and look at positive column entries to find basic var to leave
- **V2**: pick basic x_j with $\bar{b}_j<0$ to exit, and look at negative row entries to find non-basic var to enter

	Primal simplex	Dual simplex
V1	**Primal tableau**	**Dual tableau**
V2	**Dual tableau**	**Primal tableau**

Dual Simplex

- Maintain dual feasibility, terminate with dual optimality
- Rather than work in dual tableau, can track in primal tableau: perform "**dual pivots”.**

Dual Simplex

- Maintain dual feasibility, terminate with dual optimality
- Rather than work in dual tableau, can track in primal tableau: perform "**dual pivots”.**

Primal tableau properties

	Primal feasible	Dual feasible	Primal and Dual optimal
Reduced costs	Free	Non-neg	Non-neg
RHS	Non-neg	Free	Non-neg
	primal pivots to solve	dual pivots to solve	

Example: Dual Simplex

Primal problem

$$\begin{aligned} \max\ & -x_4 - x_5 \\ \text{s.t.}\ & -2x_4 - x_5 \leq 4 \\ & -2x_4 + 4x_5 \leq -8 \\ & -x_4 + 3x_5 \leq -7 \\ & x_4,\ x_5 \geq 0 \end{aligned}$$

Dual problem

$$\begin{aligned} \min\ & 4y_1 - 8y_2 - 7y_3 \\ \text{s.t.}\ & -2y_1 - 2y_2 - y_3 \geq -1 \\ & -y_1 + 4y_2 + 3y_3 \geq -1 \\ & y_1, y_2, y_3 \geq 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} \max\ & -4y_1 + 8y_2 + 7y_3 \\ \text{s.t.}\ & 2y_1 + 2y_2 + y_3 \leq 1 \\ & y_1 - 4y_2 - 3y_3 \leq 1 \\ & y_1,\ y_2,\ y_3 \geq 0 \end{aligned}$$

equivalent maximization problem

Initial Tableau

$$\begin{array}{l} \max -x_4 \quad -x_5 \\ \text{s.t. } -2x_4 \quad -x_5 \le 4 \\ \quad -2x_4 + 4x_5 \le -8 \\ \quad -x_4 + 3x_5 \le -7 \\ \quad x_4, \quad x_5 \ge 0 \end{array}$$

$$\begin{array}{l} z \qquad + x_4 + x_5 = 0 \\ \quad x_1 \quad -2x_4 - x_5 = 4 \\ \qquad x_2 \quad -2x_4 + 4x_5 = -8 \\ \qquad\quad x_3 - x_4 + 3x_5 = -7 \end{array}$$

$$\begin{array}{l} \max -4y_1 + 8y_2 + 7y_3 \\ \text{s.t. } 2y_1 + 2y_2 + y_3 \le 1 \\ \quad y_1 - 4y_2 - 3y_3 \le 1 \\ \quad y_1, \quad y_2, \quad y_3 \ge 0 \end{array}$$

$$\begin{array}{l} z + 4y_1 - 8y_2 - 7y_3 \qquad = 0 \\ \quad 2y_1 + 2y_2 + y_3 + y_4 \quad = 1 \\ \quad y_1 - 4y_2 - 3y_3 \quad + y_5 = 1 \end{array}$$

Dual Pivot (track in primal)

$$\begin{array}{l} z \qquad + x_4 + x_5 = 0 \\ \quad x_1 \quad -2x_4 - x_5 = 4 \\ \qquad x_2 \ \boxed{-2x_4} + 4x_5 = -8 \\ \qquad\quad x_3 - x_4 + 3x_5 = -7 \end{array}$$

B={1,2,3}

pivot (2,4) (2 out, 4 in)

$$\begin{array}{l} z \quad + \tfrac{1}{2}x_2 \qquad + 3x_5 = -4 \\ \quad x_1 - x_2 \qquad - 5x_5 = 12 \\ \quad -\tfrac{1}{2}x_2 \quad + x_4 - 2x_5 = 4 \\ \quad -\tfrac{1}{2}x_2 + x_3 \quad + x_5 = -3 \end{array}$$

B={1,3,4}

$$\xrightarrow[\text{pivot}]{}$$

$$\begin{array}{l} z + 4y_1 - 8y_2 - 7y_3 \qquad = 0 \\ \quad 2y_1 \boxed{+2y_2} + y_3 + y_4 \quad = 1 \\ \quad y_1 - 4y_2 - 3y_3 \quad + y_5 = 1 \end{array}$$

B={4,5}

pivot (4,2) (4 out, 2 in)

$$\begin{array}{l} z + 12y_1 \quad - 3y_3 + 4y_4 \qquad = +4 \\ \quad y_1 + y_2 + \tfrac{1}{2}y_3 + \tfrac{1}{2}y_4 \quad = \tfrac{1}{2} \\ \quad 5y_1 \quad - y_3 + 2y_4 + y_5 = 3 \end{array}$$

B={2,5}

Dual Pivot (track in primal)

$$\begin{array}{l} z \quad +\tfrac{1}{2}x_2 \qquad +3x_5 = -4 \\ x_1 - x_2 \qquad -5x_5 = 12 \\ -\tfrac{1}{2}x_2 \quad +x_4 - 2x_5 = 4 \\ \boxed{-\tfrac{1}{2}x_2} + x_3 \quad + x_5 = -3 \end{array}$$

B={1,3,4}

pivot (3,2)

$$\xrightarrow{\text{pivot}}$$

$$\begin{array}{l} z \quad + x_3 \quad + 4x_5 = -7 \\ x_1 \quad - 2x_3 \quad - 7x_5 = 18 \\ - x_3 + x_4 - 3x_5 = 7 \\ x_2 - 2x_3 \quad - 2x_5 = 6 \end{array}$$

B={1,2,4}

$$\begin{array}{l} z + 12y_1 \qquad -3y_3 + 4y_4 \qquad = +4 \\ y_1 + y_2 + \boxed{\tfrac{1}{2}y_3} + \tfrac{1}{2}y_4 \qquad = \tfrac{1}{2} \\ 5y_1 \quad - \quad y_3 + 2y_4 + y_5 = 3 \end{array}$$

B={2,5}

pivot (2,3)

$$\begin{array}{l} z + 18y_1 + 6y_2 \quad + 7y_4 = +7 \\ 2y_1 + 2y_2 + y_3 + y_4 = 1 \\ 7y_1 + 2y_2 + 3y_4 + y_5 = 4 \end{array}$$

B={3,5}

OPTIMAL!

“Upside Down Pivoting”

- Dual pivots in primal tableau. Maintain dual feasibility, terminate with dual optimality (non-negative RHS values in primal tableau).
- Primal infeasible for a while, but non-negative (primal) reduced costs. Terminate with feasible primal solution.
-
- Choose a variable with a **–ve RHS to exit**, and do ratio test on **strictly negative entries in row** with **negated reduced cost as numerator**.

Dual simplex

- Assume have a dual feasible tableau ($\bar{c}_B \geq 0$)
- **Step 1**. Pick a basic variable x_r to **leave** with a *strictly negative* **RHS** $\bar{b}_r < 0$. If no such variable then optimal and **STOP**.
- **Step 2**. Pick a nonbasic variable x_k to **enter** by considering row r and non-basic variables with **strictly negative** coefficients. *Ratio test*: k should satisfy $\mathbf{-\bar{c}_k/\bar{a}_{rk}} = \min \{\mathbf{-\bar{c}_j/\bar{a}_{rj}}: j \notin B, \bar{a}_{rj} < 0\}$.
  If all $\bar{a}_{rj}$ on $j \notin B$ nonnegative, then dual unbounded and primal infeasible and **STOP**.
- **Step 3**. *Pivot* on (r,k) and go to step 1.
- In corresponding dual tableau: basic variable y_r enters and nonbasic variable y_k leaves.

Example: A dual simplex pivot

$$z \qquad + \tfrac{1}{2}x_3 + \tfrac{1}{2}x_4 = +1\tfrac{1}{2}$$

$$x_1 \qquad + x_3 \qquad = 2$$

$$x_2 \; \boxed{-\tfrac{1}{2}x_3} + \tfrac{1}{2}x_4 = -\tfrac{1}{2}$$

(Pivot on row with –ve RHS, picking nonbasic variable to enter with strictly negative $\bar{a}_{rk}$ and minimal ratio $-\bar{c}_k / \bar{a}_{rk}$)

$$z \qquad + x_2 \qquad + x_4 = 1$$

$$x_1 + 2x_2 \qquad + x_4 = 1$$

$$-2x_2 + x_3 - x_4 = 1$$

OPTIMAL!

Example: Dual unbounded

- Suppose get to a (primal) tableau with row

  $x_1 + 2x_2 + 2x_3 = -5$
- Dual unbounded, since coefficients on x_2 and x_3 are non-negative.
- Conclude that primal is infeasible (from weak duality theorem).

Making use of the Dual Simplex

1. Find a solution after a new constraint is added (useful for sensitivity analysis, and solving IPs via branch-and-bound search)
2. Find a new solution after a change in a RHS coeff that is outside the allowable range
3. A new “Phase 1 – Phase 2” method (no need for artificial variables!)

1. Adding a New Constraint

- Recall the furniture example
- max $z = 60x_1 + 30x_2 + 20x_3$

  s.t. $8x_1 + 6x_2 + x_3 + x_4 = 48$ (lumber)

  $4x_1 + 2x_2 + 1.5x_3 + x_5 = 20$ (finishing)

  $2x_1 + 1.5x_2 + 0.5x_3 + x_6 = 8$ (carpentry)

  $x_1, \ldots, x_6 \geq 0$

  x_1 desks; x_2 tables; x_3 chairs
- $B=\{4,3,1\}$. $x^* = (2,0,8,24,0,0)$, $z=280$
- Three kinds of new constraints:
  - (i) Current optimal solution still feasible; e.g., add constraint $x_1 + x_2 + x_3 \leq 11$. Easy: just check.
  - (ii) Current basis becomes infeasible
  - (iii) LP becomes infeasible

1a. New constraint makes optimal basis infeasible.

- Add: $x_2 \geq 1$. Can just add directly to optimal tableau:
- $z \quad +5x_2 \quad +10x_5 + 10x_6 = 280$

  $-2x_2 + x_4 + 2x_5 - 8x_6 = 24$

  $-2x_2 + x_3 + 2x_5 - 4x_6 = 8$

  $x_1 + 1.25x_2 - 0.5x_5 + 1.5x_6 = 2$

  $\boxed{-x_2} \quad + x_7 = -1$
- (primal) $B=\{1,3,4,7\}$. Dual feasible.
- Dual pivot in primal (7 out, 2 in). And so pivot (7,2):

  $z \quad +10x_4 + 10x_5 \quad + 5x_7 = 275$

  $x_4 + 2x_5 - 8x_6 - 2x_7 = 26$

  $x_3 + 2x_5 - 4x_6 - 2x_7 = 10$

  $x_1 - ½x_5 + 1½x_6 + 1¼x_7 = ¾$

  $x_2 - x_7 = 1$
- Find an optimal solution: $x^* = (¾, 1, 10, 26, 0, 0)$; $z=275$.

Remark

- Could add $x_2 \geq 1$ directly to the initial tableau:
  $$x_2 - x_7 + x_8 = 1,$$
  with artificial variable x_8, and re-solve with primal simplex.
- By adding the constraint to the final tableau and using dual simplex we just need a single pivot.
- Large computational gain!

1b. New constraint makes problem infeasible.

- Consider: $x_1 + x_2 \geq 12$

$$\begin{array}{lllllllll}
z & & +5x_2 & & & +10x_5 & +10x_6 & & = 280 \\
 & & -2x_2 & & +x_4 + & 2x_5 & -8x_6 & & = 24 \\
 & & -2x_2 + x_3 + & & & 2x_5 & -4x_6 & & = 8 \\
 & x_1 & +1.25x_2 & & & -0.5x_5 & +1.5x_6 & & = 2 \quad (4) \\
 & -x_1 & -x_2 & & & & & +x_7 & = -12 \quad (5)
\end{array}$$

- x_1 no longer isolated. Adopt: (5'):=(4)+(5)
  $$0.25x_2 \boxed{-0.5x_5} + 1.5x_6 + x_7 = -10 \qquad (5')$$
- (Primal) B={4,3,1,7}. Dual feasible.
- Dual pivot in primal: (7 out, 5 in).
- Get tableau on next slide.

1b. New constraint makes problem infeasible.

$$z \quad +10x_2 \qquad\qquad +40x_6 +20x_7 = 80$$
$$-x_2 \quad +x_4 \qquad -2x_6 + 4x_7 = -16$$
$$\rightarrow \quad \boxed{-x_2} + x_3 + \qquad 2x_6 + 4x_7 = -32$$
$$x_1 + x_2 \qquad\qquad - x_7 = 12$$
$$-½x_2 + \qquad x_5 - 3x_6 - 2x_7 = 12$$
$$-½x_2 + \qquad x_5 - 3x_6 - 2x_7 = 20$$

- (Primal) B={4,3,1,5}
- Dual pivot in primal: 3 out, 2 in
- Get tableau on next slide

1b. New constraint makes problem infeasible.

$$z \qquad +10x_3 \qquad +60x_6 +60x_7 = -240$$
$$-x_3 +x_4 \qquad -4x_6 \qquad = -16$$
$$x_2 - x_3 \qquad -2x_6 \quad -4x_7 = 32$$
$$\boxed{x_1 \qquad +x_3 \qquad +2x_6 \quad +3x_7 = -20}$$
$$-½x_3 \qquad +x_5 - 4x_6 \quad -4x_7 = 36$$

- (Primal) B={4,2,1,5}
- Dual pivot in primal: “1 out.” But ratio test fails, no variable to enter.
- Dual unbounded, and primal infeasible (by weak duality).

Making use of the Dual Simplex

1. Find a solution after a new constraint is added (useful for sensitivity analysis, and solving IPs via branch-and-bound search)
2. Find a new solution after a change in a RHS coeff that is outside the allowable range
3. A new “Phase 1 – Phase 2” method (no need for artificial variables!)

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2. Changing a RHS

- Recall BFS for B={1,3,4} in furniture problem remains feasible for $16 \leq b_2 \leq 24$. Suppose $b_2 := 30$.
- Let’s consider what happens at same basis.
- $z = y^T b = (0\ 10\ 10) \begin{pmatrix} 48 \\ 30 \\ 8 \end{pmatrix} = 380$

$$\bar{b} = A_B^{-1} b = \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix} \begin{pmatrix} 48 \\ 30 \\ 8 \end{pmatrix} = \begin{pmatrix} 44 \\ 28 \\ -3 \end{pmatrix}$$

- Construct tableau. Next slide.

- $z \quad +5x_2 \quad +10x_5 + 10x_6 = 380$

  $-2x_2 + x_4 + 2x_5 - 8x_6 = 44$

  $-2x_2 + x_3 + 2x_5 - 4x_6 = 28$

  $x_1 + 1.25x_2 \quad \boxed{-0.5x_5} + 1.5x_6 = -3$

  $B=\{1,3,4\}$. Primal infeasible basis. Do a dual pivot in primal (1 out, 5 in.)

$z + 20x_1 + 30x_2 \quad +40x_6 = 320$

$4x_1 + 3x_2 + x_4 - 2x_6 = 32$

$4x_1 + 3x_2 + x_3 + 2x_6 = 16$

$-2x_1 - 2\frac{1}{2}x_2 + x_5 - 3x_6 = 6$

$B=\{3,4,5\}$

OPTIMAL! Solution $x^*=(0,0,16,32,6,0)$. Only make chairs!

Making use of the Dual Simplex

1. Find a solution after a new constraint is added (useful for sensitivity analysis, and solving IPs via branch-and-bound search)
2. Find a new solution after a change in a RHS coeff that is outside the allowable range
3. A new “Phase 1 – Phase 2” method (no need for artificial variables!)

3. New "Phase 1 - Phase 2" method

(a) $\bar{b} \geq 0$: use **primal simplex** (no phase 1)

(b) $\bar{c} \geq 0$, some $\bar{b}_i < 0$: use **dual simplex** (no phase 1)

(c) Some $\bar{b}_i < 0$ and some $\bar{c}_j < 0$

- E.g., suppose "max x_1 -2x_2."
- First solve "max –x_1 -2x_2." This has $z+x_1 +2x_2$ first row, and so dual feasible. **Use dual simplex** to find a primal feasible solution.
- Second, modify tableau to introduce correct objective. **Use primal simplex**.

- ALTERNATIVE: Replace RHS $\bar{b}$ with $\bar{b} \geq 0$, use primal simplex to find a dual feasible solution, then bring back correct RHS and use dual simplex.

Summary: Dual Simplex

- **Dual simplex:** simplex method applied to the dual problem. Maintains a **dual feasible basis, terminates with an optimal dual basis**.
- We **track the dual simplex on primal tableau**. "Upside down pivots": first which basic var exits (-ve RHS), then which non-basic enters?
- Flexible approach: can work in primal tableau and move seamlessly from primal to dual pivots.
- Useful when constraints change, for phase1-phase 2, and we'll use for solving IPs!