

AM 121 Introduction to Optimization: Models and Methods
Example Questions for Midterm 1

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Here are some practice questions to help to prepare for the midterm. The midterm will not contain as many questions.

1. (Standard forms)

- (a) Convert the following problem into *standard inequality form*. (Hint, the standard form is a *maximization*.)

$$\begin{aligned} \min \quad & 2x_1 - 4x_2 \\ \text{s.t.} \quad & 5x_1 - 3x_2 \geq -1 \\ & 2x_1 \leq 7 \\ & 4.5x_1 + 2x_2 = 20 \\ & x_1, x_2 \geq 0 \end{aligned}$$

- (b) Convert the following problem into *standard equality form*. Be sure to note any *slack*, *excess* or *artificial* variables that you introduce. Introduce as few additional variables as possible.

$$\begin{aligned} \min \quad & 3x_1 - x_2 + 5x_3 \\ \text{s.t.} \quad & x_1 - 2x_2 \geq 4 \\ & 2x_1 + x_3 \geq 2 \\ & 7x_1 - 2x_2 + 5x_3 = -5 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

2. (Simplex method)

Solve the following linear program using the simplex method. Do this by hand (since you will have to on the actual midterm). Choose as an entering variable the one with the lowest reduced cost.

$$\begin{aligned} \max \quad & 2x_1 - x_2 + x_3 \\ \text{s.t.} \quad & 3x_1 + x_2 + 2x_3 \leq 70 \\ & x_1 - x_2 + 2x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

3. (Conceptual)

- (a) Consider the following tableau for a maximization problem. Give the weakest conditions on the unknowns a_1, a_2, a_3, b and c (perhaps negative) that make the following statements true by inspection of the tableau:
- (i) The current basic solution is optimal.
 - (ii) The current basic solution is optimal and there are alternative optimal solutions.
 - (iii) The LP is unbounded.
 - (iv) The current basic solution is infeasible.
 - (v) The current basic solution is feasible but the objective value can be improved by bringing x_1 into the basis and removing x_4 .

$$\begin{array}{rcccccccl} z & +cx_1 & +2x_2 & & & & & = & 10 \\ & -x_1 & +a_1x_2 & +x_3 & & & & = & 4 \\ & a_2x_1 & -4x_2 & & +x_4 & & & = & 1 \\ & a_3x_1 & +3x_2 & & & +x_5 & & = & b \end{array}$$

(b) Fill in the blanks:

- (i) We should expect there to be less rows than columns in an linear program in standard equality form because _____

- (ii) The convexity of the polyhedron that corresponds to the feasible solution space for a linear program is a crucial property in being able to solve linear programs efficiently because _____

4. (Graphical sensitivity) Consider the LP

$$\begin{array}{ll} \max & 4x_1 + x_2 \\ \text{s.t.} & 3x_1 + x_2 \leq 6 \\ & 5x_1 + 3x_2 \leq 15 \\ & x_1, x_2 \geq 0 \end{array}$$

The optimal solution is $z = 8, x_1 = 2, x_2 = 0$. Use the graphical approach to answer the following questions:

- (a) Determine the range of values of c_1 (coefficient of x_1) for which the current basis remains optimal.

- (b) Determine the range of values of c_2 (coefficient of x_2) for which the current basis remains optimal.
- (c) Determine the range of values of b_1 (RHS of first constraint) for which the current basis remains optimal.
- (d) Determine the range of values of b_2 (RHS of second constraint) for which the current basis remains optimal.

5. (Quick fire) Just answer ‘true’ or ‘false.’ No need to explain.

1. True or False: For an LP to be unbounded, the LP’s feasible region must be unbounded.
2. True or False: A tableau is dual feasible when the reduced costs are all nonnegative.
3. True or False: The dual simplex method (when used on the primal tableau) first chooses a variable to enter.
4. True or False: The dual simplex method is useful because it can typically recover in only a few pivots an optimal solution to an LP that has been slightly modified by introducing a new constraint.
5. True or False: The optimal dual value that corresponds to a binding primal constraint in an optimal primal solution necessarily has “no slack” (is non-zero).
6. True or False: Bland’s theorem ensures that no variable will enter a tableau more than once when the smallest subscript rule is used for pivoting.

6. (Modeling)

You have decided to enter the candy business. You are considering producing two types of candies: Slugger Candy and Easy Out Candy, both of which consist solely of sugar, nuts and chocolate. At present, you have in stock 100 oz of sugar, 20 oz of nuts, and 30 oz of chocolate. The mixture used to make Easy Out Candy must contain at least 20% nuts. The mixture used to make Slugger Candy must contain at least 10% nuts and 10% chocolate. Each ounce of Easy Out can be sold for 25¢ and each ounce of Slugger for 20¢. Formulate a *mathematical model for an LP* that will enable you to maximize your revenue from candy sales. Describe the elements of your model.

7. (Duality)

Consider an LP with two decision variables $x_1, x_2 \geq 0$ and an objective function

$$\max 2x_2$$

- (a) Define two inequality (\leq) constraints, each involving both x_1 and x_2 , such that the LP is unbounded.
- (b) Define the dual of the problem you formulate in (a) and plot the feasible region to show that it is infeasible.
- (c) Use one of the duality theorems to explain why primal unbounded implies dual infeasible.

8. (Duality)

- (a) Use the duality definition for the *standard inequality form* to find the dual of the following:

$$\begin{aligned} \max \quad & 4x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & 2x_1 + x_2 + x_3 \leq -1 \\ & x_1 + x_2 + 2x_3 \leq 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

- (b) Transform the resulting program into a *maximization* problem and again find the dual.
(c) What do you notice?

9. (Sensitivity with AMPL)

[The AMPL file will be posted with the solution.]

Zales Jewelers uses rubies and sapphires to produce two types of rings. A Type 1 ring requires 2 rubies, 4 sapphires, and 1 hour of labor. A Type 2 ring requires 4 rubies, 2 sapphires, and 2 hours of labor. Each Type 1 ring sells for \$400; type 2 sells for \$500. At present, Zales has 100 rubies, 120 sapphires, and 70 hours of labor. Extra rubies can be purchased at a cost of \$140 per ruby.

Market demand requires that the company produce at least 15 Type 1 rings and at least 16 Type 2. Let x_1 = number of Type 1 rings produced, x_2 = number of Type 2 rings produced, and r denote the number of rubies purchased. To maximize revenue, Zales solves the following LP:

$$\begin{aligned} \max \quad & 400x_1 + 500x_2 - 140r \\ \text{s.t.} \quad & 2x_1 + 4x_2 - r \leq 100 & (1) \\ & 4x_1 + 2x_2 \leq 120 & (2) \\ & x_1 + 2x_2 \leq 70 & (3) \\ & x_1 \geq 15 & (4) \\ & x_2 \geq 16 & (5) \\ & x_1, x_2, r \geq 0 \end{aligned}$$

An LP has been formulated in AMPL and this is the solution, with corresponding objective value:

$$x_1 = 22; \quad x_2 = 16; \quad r = 8; \quad z = 15,680.$$

The following sensitivity information is generated:

```
: _varname _var.rc _var.down _var.current _var.up :=
1  x1          0          280          400          1e+20
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2	x2	0	-1e+20	500	620	
3	r	0	-200	-140	-100 ;	
:						
	_conname	_con.dual	_con.down	_con.current	_con.up	:=
1	constraint1	140	-1e+20	100	108	
2	constraint2	30	104	120	184	
3	constraint3	0	54	70	1e+20	
4	constraint4	0	-1e+20	15	22	
5	constraint5	-120	13.3333	16	26.6667 ;	

Answer the following questions:

- Suppose that instead of \$100, each ruby costs \$190. Would Zales still purchase rubies? What would be the new optimal solution to the problem?
- Suppose that Zales were only required to produce at least 14 Type 2 rings. What would Zales' revenue be?
- What is the most that Zales would be willing to pay for another hour of labor?
- What is the most that Zales would be willing to pay for another sapphire?
- Zales is considering producing Type 3 rings. Each Type 3 ring can be sold for \$550 and requires 4 rubies, 2 sapphires and 1 hour of labor. Should Zales produce any Type 3 rings? (Hint: you will need to do a small amount of hand calculation here)

10. (Degeneracy)

- Describe the condition that makes a feasible tableau degenerate.
- Consider the following tableau for a maximization problem. Write down the current basis, current basic solution and current objective value.

$$\begin{array}{rcccccc}
 z & +\frac{1}{2}x_1 & -2x_2 & & +\frac{3}{2}x_4 & = & 3 \\
 & +\frac{1}{2}x_1 & & +x_3 & +\frac{1}{2}x_4 & = & 1 \\
 & -x_1 & +x_2 & & -x_4 & +x_5 & = & 0
 \end{array}$$

- Perform two pivots. After each pivot write down the new basis, the new basic solution, and the new objective value. When choosing an entering variable, select the one with the smallest reduced cost. What do you notice?
- What is the general problem that can be caused by degeneracy in the simplex method and what is a solution?

11. (Pivoting)

Consider the following tableau for a maximization problem:

$$\begin{array}{rcccccc} z & & -2x_2 & +2x_3 & & -3x_5 & = & 5 \\ & & +2x_2 & -x_3 & & -x_5 & +x_6 & = & 4 \\ & & +x_2 & -x_3 & +x_4 & +x_5 & & = & 2 \\ +x_1 & -2x_2 & -2x_3 & & & -3x_5 & & = & 6 \end{array}$$

- (a) List all pairs (x_r, x_k) such that x_k could be the entering variable and x_r could be the leaving variable.
- (b) List all pairs if the “most negative reduced cost” rule for choosing the entering variable is used.
- (c) List all pairs if the smallest subscript rule is used for choosing the entering and leaving variables.

12. (Sensitivity)

Consider the following LP and its optimal tableau (where x_4 and x_5 are the slack variables for the two inequalities):

$$\begin{array}{ll} \max & 3x_1 + x_2 - x_3 \\ \text{s.t.} & 2x_1 + x_2 + x_3 \leq 8 \\ & 4x_1 + x_2 - x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$$\begin{array}{rcccccc} z & & +x_3 & +\frac{1}{2}x_4 & +\frac{1}{2}x_5 & = & 9 \\ +x_1 & & -x_3 & -\frac{1}{2}x_4 & +\frac{1}{2}x_5 & = & 1 \\ +x_2 & +3x_3 & +2x_4 & -x_5 & & = & 6 \end{array}$$

Recall the following algebra relating an original problem in standard equality form and the tableau for basis B :

$$\begin{aligned} \bar{b} &= A_B^{-1}b \\ y^T &= c_B^T A_B^{-1} \\ \bar{c}_j &= c_B^T A_B^{-1} A_j - c_j = y^T A_j - c_j, \quad \forall j \in B' \end{aligned}$$

Here, the optimal basis is $B = \{1, 2\}$ and

$$A_B = \begin{pmatrix} 2 & 1 \\ 4 & 1 \end{pmatrix} \quad A_B^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 2 & -1 \end{pmatrix}$$

Using the optimal tableau or these equations, answer the following questions:

- (a) What is the optimal solution to the dual of this LP?
- (b) Find the range of values of b_2 (the RHS on the second constraint) for which the current basis remains optimal. If $b_2 = 12$ what is the new optimal solution?
- (c) Find the range of values of c_3 (the objective function coefficient on the third variable) for which the current solution remains optimal.
- (d) Find the range of values of c_1 (the objective function coefficient on the first variable) for which the current basis remains optimal. **(Note: it is more important you understand how to solve this problem than to actually solve it here. It's good algebra practice, so we won't stop you, but don't expect us to give this much algebra on the midterm itself.)**

13. (Dual simplex)

In solving the following LP

$$\begin{aligned}
 \max \quad & 6x_1 + x_2 \\
 \text{s.t.} \quad & x_1 + x_2 \leq 5 \\
 & 2x_1 + x_2 \leq 6 \\
 & x_1, x_2 \geq 0
 \end{aligned}$$

we obtain the optimal tableau

$$\begin{array}{rcccc}
 z & & +2x_2 & & +3x_4 & = & 18 \\
 & & +\frac{1}{2}x_2 & +x_3 & -\frac{1}{2}x_4 & = & 2 \\
 +x_1 & & +\frac{1}{2}x_2 & & +\frac{1}{2}x_4 & = & 3
 \end{array}$$

where x_3 and x_4 are the slack variables introduced for the two inequalities.

- (a) Find the optimal solution if we add the constraint $3x_1 + x_2 \leq 10$ to the original LP.
- (b) Find the optimal solution if we add the constraint $x_1 - x_2 \geq 6$ to the original LP.
- (c) Find the optimal solution if we add the constraint $8x_1 + x_2 \leq 12$ to the original LP.