AM 121 Introduction to Optimization: Models and Methods
Solution to Example Questions for Midterm 1

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Fall 2016

Here are some practice questions to help to prepare for the midterm. The midterm will not contain as many questions.

1. (Standard forms)

(a) Convert the following problem into *standard inequality form.* (Hint, the standard form is a *maximization.*)

$$\begin{aligned} \min \quad & 2x_1 - 4x_2 \\ \text{s.t.} \quad & 5x_1 - 3x_2 \geq -1 \\ & 2x_1 \leq 7 \\ & 4.5x_1 + 2x_2 = 20 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(b) Convert the following problem into *standard equality form.* Be sure to note any *slack*, *excess* or *artificial* variables that you introduce. Introduce as few additional variables as possible.

$$\begin{aligned} \min \quad & 3x_1 - x_2 + 5x_3 \\ \text{s.t.} \quad & x_1 - 2x_2 \geq 4 \\ & 2x_1 + x_3 \geq 2 \\ & 7x_1 - 2x_2 + 5x_3 = -5 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution to task

(a) The first thing is to turn the minimization problem into a maximization problem by multiplying the objective by -1. This leads to the equivalent objective:

$$\max \quad -2x_1 + 4x_2$$

We have to transform the first constraint from $\geq$ to $\leq$, which we can do again by multiplying by -1:

$$-5x_1 + 3x_2 \leq 1$$

Since the second constraint is already in standard inequality form, nothing has to be done here. The equality constraint can be expressed as a pair of $\le$ constraints:

$$\begin{aligned} 4.5x_1 + 2x_2 &\le 20 \\ -4.5x_1 - 2x_2 &\le -20. \end{aligned}$$

We obtain the following LP in standard inequality form:

$$\begin{aligned} \max \quad & -2x_1 + 4x_2 \\ \text{s.t.} \quad & -5x_1 + 3x_2 \le 1 \\ & 2x_1 \le 7 \\ & 4.5x_1 + 2x_2 \le 20 \\ & -4.5x_1 - 2x_2 \le -20 \\ & x_1, x_2 \ge 0 \end{aligned}$$

(b) We can again turn the minimization problem into an equivalent maximization problem by multiplying the objective function by -1. For the first and second constraints, we introduce *excess* variables x_4 and x_5 and rewrite the constraints as equalities:

$$\begin{aligned} x_1 - 2x_2 - x_4 &= 4 \\ 2x_1 + x_3 - x_5 &= 2 \end{aligned}$$

The third equation is already in equality form with a negative value in the right hand side vector. By definition of standard equality form, we can allow for b to be any sign. Summarizing everything we obtain the following optimization problem in standard equality form:

$$\begin{aligned} \max \quad & -3x_1 + x_2 - 5x_3 \\ \text{s.t.} \quad & x_1 - 2x_2 - x_4 = 4 \\ & 2x_1 + x_3 - x_5 = 2 \\ & 7x_1 - 2x_2 + 5x_3 = -5 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

End Solution

2. (Simplex method)

Solve the following linear program using the simplex method. Do this by hand (since you will have to on the actual midterm). Choose as an entering variable the one with the lowest reduced cost.

$$\begin{aligned} \max \quad & 2x_1 - x_2 + x_3 \\ \text{s.t.} \quad & 3x_1 + x_2 + 2x_3 \le 70 \\ & x_1 - x_2 + 2x_3 \le 10 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Solution to task

We first convert the problem to standard equality form by introducing slack variables x_4 and x_5 on the constraints. Note that the slack variables can serve as our initial basis and that we are already in canonical form. Furthermore, we will write the objective value as z:

$$\begin{aligned} z - 2x_1 + x_2 - x_3 &= 0 \\ 3x_1 + x_2 + 2x_3 + x_4 &= 70 \\ x_1 - x_2 + 2x_3 + x_5 &= 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Writing this in tableau form:

$$\begin{bmatrix} var & z & x_1 & x_2 & x_3 & x_4 & x_5 & RHS \\ z & 1 & -2 & 1 & -1 & 0 & 0 & 0 \\ x_4 & 0 & 3 & 1 & 2 & 1 & 0 & 70 \\ x_5 & 0 & 1 & -1 & 2 & 0 & 1 & 10 \end{bmatrix}$$

Here $x = (0, 0, 0, 70, 10)$ and $z = 0$. Since x_1 has the smallest reduced cost, we have it enter. By the ratio test, x_5 has the lowest ratio and thus should leave. We perform the pivot:

$$\begin{bmatrix} var & z & x_1 & x_2 & x_3 & x_4 & x_5 & RHS \\ z & 1 & 0 & -1 & 3 & 0 & 2 & 20 \\ x_4 & 0 & 0 & 4 & -4 & 1 & -3 & 40 \\ x_1 & 0 & 1 & -1 & 2 & 0 & 1 & 10 \end{bmatrix}$$

Here $x = (10, 0, 0, 40, 0)$ and $z = 20$. Since x_2 is the only variable with negative reduced cost, we enter on x_2. Since only x_4 has a positive in that column, x_4 leaves. We perform the pivot:

$$\begin{bmatrix} var & z & x_1 & x_2 & x_3 & x_4 & x_5 & RHS \\ z & 1 & 0 & 0 & 2 & 1/4 & 5/4 & 30 \\ x_2 & 0 & 0 & 1 & -1 & 1/4 & -3/4 & 10 \\ x_1 & 0 & 1 & 0 & 1 & 1/4 & 1/4 & 20 \end{bmatrix}$$

Here $x = (20, 10, 0, 0, 0)$ and $z = 30$. Since all reduced costs are nonnegative, we have reached an optimal solution.

We used the following Maple commands (aka, we cheated):

```
with(linalg);
A := <<1,0,0>|<-2,3,1>|<1,1,-1>|<-1,2,2>|<0,1,0>|<0,0,1>|<0,70,10>>;
SideVars := vector(3, [z, x_4, x_5]);
Vars := vector(8, [var, z, x_1, x_2, x_3, x_4, x_5, RHS]);
Disp := concat(SideVars,A);
Disp := stackmatrix(Vars,Disp);

A := pivot(A, 3,2);
```

```
SideVars := vector(3, [z, x_4, x_1]);
Disp := concat(SideVars,A);
Disp := stackmatrix(Vars,Disp);

A := pivot(A, 2,3);
A := mulrow(A, 2, 1/A[2,3]);
SideVars := vector(3, [z, x_2, x_1]);
Disp := concat(SideVars,A);
Disp := stackmatrix(Vars,Disp);
```

End Solution

3. (Conceptual)

(a) Consider the following tableau for a maximization problem. Give the weakest conditions on the unknowns a_1, a_2, a_3, b and c (perhaps negative) that make the following statements true by inspection of the tableau:

(i) The current basic solution is optimal.

(ii) The current basic solution is optimal and there are alternative optimal solutions.

(iii) The LP is unbounded.

(iv) The current basic solution is infeasible.

(v) The current basic solution is feasible but the objective value can be improved by bringing x_1 into the basis and removing x_4.

$$\begin{array}{cccccccc} z & +cx_1 & +2x_2 & & & & = & 10 \\ & -x_1 & +a_1x_2 & +x_3 & & & = & 4 \\ & a_2x_1 & -4x_2 & & +x_4 & & = & 1 \\ & a_3x_1 & +3x_2 & & & +x_5 & = & b \end{array}$$

Solution to task

(a) (i) We need $b \geq 0$ for feasibility (which is required for optimality). For optimality, we need $c \geq 0$.

(ii) We need to be able to pivot from an optimal tableau without changing the objective value. This requires $c = 0$, and to be able to pivot in, we need $b > 0$ and $a_2 > 0$ or $a_3 > 0$.

(iii) The problem is unbounded if we can increase x_1 without bound to increase the value of the objective. This requires the current solution to be feasible ($b \geq 0$), $c < 0$, and all coefficients in the x_1 columns to be less than or equal to 0, that is, $a_2 \leq 0$ and $a_3 \leq 0$.

(iv) $b < 0$ would give us an infeasible solution, as x_5 would violate the non-negativity condition.

(v) For feasibility, we need $b \geq 0$. To bring in x_1 to improve the objective, we need negative reduced cost $c < 0$. For x_4 to leave, we need $a_2 > 0$. Furthermore, we either need to have $a_3 \leq 0$ (in which case x_4 is the only candidate for leaving), or x_4 must have a weakly lower ratio than x_5, that is, $\frac{1}{a_2} \leq \frac{b}{a_3}$ (in particular we need $b > 0$ if $a_3 > 0$).

End Solution

(b) Fill in the blanks:

(i) We should expect there to be less rows than columns in an linear program in standard equality form because ____

____

(ii) The convexity of the polyhedron that corresponds to the feasible solution space for a linear program is a crucial property in being able to solve linear programs efficiently because ____

____

Solution to task

(b) Fill in the blanks:

(i) We can assume wlog that the rows are linearly independent (else there is redundancy.) Implies $m \leq n$ (since row rank $\leq$ column rank.) In particular, must have $m < n$. Otherwise, $m = n$ and there are m equations and n variables and at most one solution, and would be no degree of freedom for any optimization.

(ii) An extreme point is guaranteed to be globally optimal as long as its associated objective value is the highest among all neighboring extreme points.

End Solution

4. (Graphical sensitivity) Consider the LP

$$\begin{aligned} \max \quad & 4x_1 + x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \leq 6 \\ & 5x_1 + 3x_2 \leq 15 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The optimal solution is $z = 8, x_1 = 2, x_2 = 0$. Use the graphical approach to answer the following questions:

(a) Determine the range of values of c_1 (coefficient of x_1) for which the current basis remains optimal.

(b) Determine the range of values of c_2 (coefficient of x_2) for which the current basis remains optimal.

(c) Determine the range of values of b_1 (RHS of first constraint) for which the current basis remains optimal.

(d) Determine the range of values of b_2 (RHS of second constraint) for which the current basis remains optimal.

Solution to task

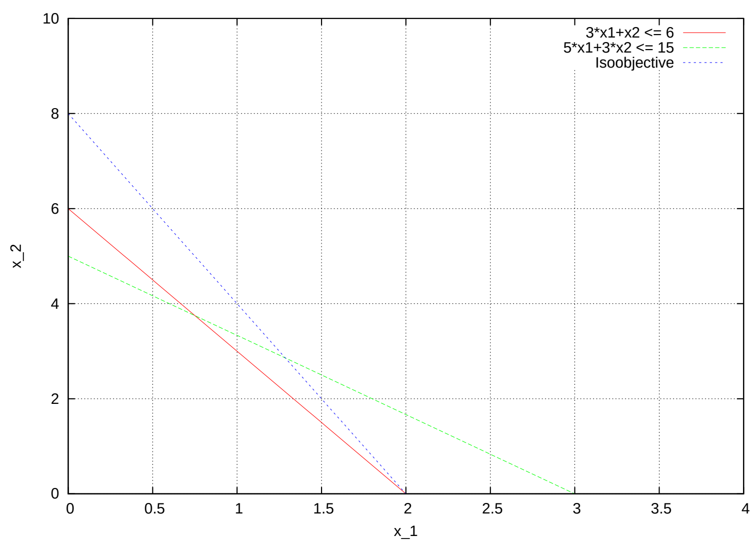

Figure 1: Constraints and iso-objective function

Fig. 1 shows the two constraints and the iso-objective function. The optimal solution is at the lower right corner of the feasible region.

(a) Note that if c_1 were negative and the isoprofit line had positive slope, the isoprofit contour lines will move up and to the left, in which case the basis is no longer optimal. For positive c_1, note that as long as the slope of the objective function remains smaller (in this case, more negative) than the slope of the first constraint, the basis will not change. Since the slope of the first constraint is -3 and the slope of the objective is $-c1/c2$ with $c_2 = 1$, c_1 must satisfy:

$$\begin{aligned} -3 &\geq -\tfrac{c_1}{1} \\ \Rightarrow \quad 3 &\leq c_1 \end{aligned}$$

(b) We can apply the same reasoning, with $c_1 = 4$. Note that if c_2 were nonpositive, the isoprofit contour lines increases as they shift to the right, thus maintaining the current basis as optimal. When c_2 is postive, we must satisfy:

$$\begin{aligned} -3 &\geq -\tfrac{4}{c_2} \\ \Rightarrow \quad 3/4 &\leq \tfrac{1}{c_2} \\ \Rightarrow \quad c_2 &\leq 4/3 \end{aligned}$$

Thus any $c_2 \leq 4/3$ is within range.

(c) The right hand side element b_1 controls the distance of the first constraint from the origin. By increasing b_1, the line representing the first constraint shifts to the right. Once this line crosses the x_1 axis beyond where the line representing the second constraint crosses the x_1 axis, the second constraint becomes binding and the basic feasible solution changes. From the graph we see that the second constraint crosses the x_1 axis at $x_1 = 3$. The first constraint crosses at $x_1 = b_1/3$. Hence we must have

$$b_1/3 \leq 3 \quad \Rightarrow \quad b_1 \leq 9$$

As a lower bound, note that we must have $b_1 \leq 0$ to maintain feasibility. This gives us the range:

$$0 \leq b_1 \leq 9$$

(d) We can use the same argument as in the previous subtask. This time the point of interest is where the intersection of the second constraint with the x_1 axis moves so far to the left that it becomes binding, which is when it intersects to the left of $x_1 = 2$. This corresponds to the condition

$$2 \leq b_2/5 \quad \Rightarrow 10 \leq b_2$$

Note there is no upper bound on b_2 as it continues to not bind as it shifts to the right. Thus we have:

$$10 \leq b_2$$

As a sanity check, we implemented an AMPL model and looked at its sensitivity report:

```
var x1 >= 0;
var x2 >= 0;

maximize objective: 4*x1 + x2;

subject to constraint1: 3*x1 + x2  <= 6;
subject to constraint2: 5*x1 + 3*x2 <= 15;
---------------------
reset;
reset data;

model graphsensitivity.mod;

option cplex_options 'sensitivity primalopt presolve 0';
option presolve 0;
solve;

display x1; display x2;

display _varname, _var.rc, _var.down, _var.current, _var.up > graphsensitivity.sens;
display _conname, _con.dual, _con.down, _con.current, _con.up > graphsensitivity.sens;
----------------------
: _varname    _var.rc  _var.down _var.current     _var.up      :=
1   x1        0               3        4        1e+20
2   x2       -0.333333   -1e+20        1            1.33333
;

:    _conname   _con.dual _con.down _con.current _con.up    :=
1   constraint1   1.33333      0            6          9
2   constraint2   0           10           15       1e+20
;
```

We see that we have got the same solution.

End Solution

5. (Quick fire) Just answer 'true' or 'false.' No need to explain.

1. True or False: For an LP to be unbounded, the LP's feasible region must be unbounded.
2. True or False: A tableau is dual feasible when the reduced costs are all nonnegative.
3. True or False: The dual simplex method (when used on the primal tableau) first chooses a variable to enter.

4. True or False: The dual simplex method is useful because it can typically recover in only a few pivots an optimal solution to an LP that has been slightly modified by introducing a new constraint.

5. True or False: The optimal dual value that corresponds to a binding primal constraint in an optimal primal solution necessarily has "no slack" (is non-zero).

6. True or False: Bland's theorem ensures that no variable will enter a tableau more than once when the smallest subscript rule is used for pivoting.

Solution to task (We explain why, you wouldn't need to.)

1. True. To have the objective value go off to infinity, some variable value must be going off to infinity. For this to be possible, the feasible region must be unbounded.

2. True. A dual feasible tableau maintains 'optimal' reduced costs $\bar{c} \geq 0$, whereas a primal feasible tableau maintains feasibility ($\bar{b} \geq 0$).

3. False! It first chooses a variable to leave (by consulting RHS values).

4. True. From the optimal solution we can easily arrive at a dual feasible basic solution, from which by applying dual simplex we can usually quickly find a new optimal solution.

5. False! From complementary slackness, we know $y_i s_i = 0$ for all constraints i. If $s_i = 0$, y_i can take on any value (since constraint has no slack). [CS insists that AT MOST one has slack.]

6. False! Bland's theorem ensures that no tableau will appear more than once, but can allow a variable to enter twice.

End Solution

6. (Modeling)

You have decided to enter the candy business. You are considering producing two types of candies: Slugger Candy and Easy Out Candy, both of which consist solely of sugar, nuts and chocolate. At present, you have in stock 100 oz of sugar, 20 oz of nuts, and 30 oz of chocolate. The mixture used to make Easy Out Candy must contain at least 20% nuts. The mixture used to make Slugger Candy must contain at least 10% nuts and 10% chocolate. Each ounce of Easy Out can be sold for 25¢ and each ounce of Slugger for 20¢. Formulate a *mathematical model for an LP* that will enable you to maximize your revenue from candy sales. Describe the elements of your model.

Solution to task

Note: we provide a general model with parameters for the data. Your solution during the midterm would not need to be general (it could include numbers

directly) but the essential elements of your model should be described. (More briefly than what we do here!)

Set Definitions

- Let $i \in I$ be the ingredients used to make candy
- Let $c \in C$ be the types of candy we make

Parameter Definitions

- Let s_i be the stock we have of each ingredient in ounces
- Let r_c be the revenue we get from each ounce of candy sold
- Let p_{ic} be the minimum amount of ingredient i in candy c

Variable Definitions

- Let x_{ic} be a decision variable specifying the ounces of ingredient i used to make candy c

Objective Definition

The amount of a given candy we sell, is the sum of the amount of ingredients we use to produce that candy. (Even the matter in candy is neither created nor distroyed without nuclear processes). For a given candy c our revenue in cents should then be $r_c \sum_{i \in I} x_{ic}$ We want to maximize the amount of revenue we get from all the candy we make, so we get: $\sum_{c \in C} r_c \sum_{i \in I} x_{ic}$ we can pull the constant into the sum if we like, giving us:

$$\max \sum_{c \in C} \sum_{i \in I} r_c x_{ic}$$

Constraint Definitions

First we need a set of constraints to ensure the amount of each ingredient used is less then or equal to the amount available:

$$\sum_{c \in C} x_{ic} \leq s_i \quad \forall i \in I$$

Now for the tricky part. We need to ensure that fraction of each ingredient in each product, is at least the minimum amount required:

$$\frac{x_{ic}}{\sum_{i' \in I} x_{i'c}} \geq p_{ic} \quad \forall i \in I \text{ and } c \in C$$

But these constraints are not linear. We can however linearize them with a little algebra:

$$x_{ic} \geq p_{ic} \sum_{i' \in I} x_{i'c} \quad \forall i \in I \text{ and } c \in C$$

Or rearranging:

$$x_{ic} - p_{ic} \sum_{i' \in I} x_{i'c} \geq 0 \quad \forall i \in I \text{ and } c \in C$$

Instantiating Data

Each of the above sets and parameters is easily instantiated from the information in the problem statement:

- $I = \{\text{Sugar, Nuts, Chocolate}\}$
- $C = \{\text{Slugger, EasyOut}\}$
- $s = (100, 20, 30)$
- $r = (20, 25)$
- $p = \begin{vmatrix} 0 & 0 \\ .1 & .2 \\ .1 & 0 \end{vmatrix}$

End Solution

7. (Duality)

Consider an LP with two decision variables $x_1, x_2 \geq 0$ and an objective function

$$\max \ 2x_2$$

(a) Define two inequality ($\leq$) constraints, each involving both x_1 and x_2, such that the LP is unbounded.

(b) Define the dual of the problem you formulate in (a) and plot the feasibile region to show that it is infeasible.

(c) Use one of the duality theorems to explain why primal unbounded implies dual infeasible.

Solution to task

(a) Since x_2 has a positive coefficient in the objective, as long as our constraints allow x_2 to extend in the positive direction the LP will be unbounded. Consider:

$$x_1 - x_2 \leq 6$$
$$2x_1 - x_2 \leq 4$$

To see the LP is unbounded, fix $x_1 = 0$. Letting $x_2 \to \infty$ satisfies the constraints and leads to an unbounded solution.

(b) We associate dual variables y_1 and y_2 with the two constraints. We have the following dual formulation:

$$\begin{aligned} \min \quad & 6y_1 + 4y_2 \\ \text{s.t.} \quad & y_1 + 2y_2 \geq 0 \\ & -y_1 - y_2 \geq 2 \\ & y_1, y_2 \geq 0 \end{aligned}$$

We obtain the following plot (using Maple here... yes, we cheated):

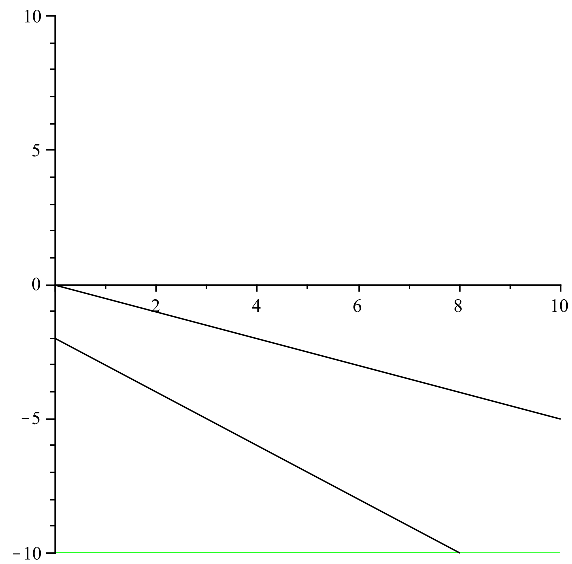

Figure 2: Infeasible region

The feasible region must be above the top line and below the bottom line. Clearly this is infeasible. Further note that because of non-negativity constraints, the second constraint already ensures the problem is infeasible (no points in positive region for both y_1 and y_2). **In your diagrams, be sure to label the directions of inequalities (say, with arrows) to illustrate that the region is indeed infeasible!**

We made use of the following Maple commands (you don't need to know this for the midterm– this is just FYI and for completeness! Same goes for Maple commands for pivoting):

```
with(plots);
constraints := {y[1] + 2*y[2] >= 0, -1*y[1] - y[2] >= 2, y[1]>=0, y[2]>=0};
domain:=inequal(constraints,y[1]=0..10,y[2]=-10..10,
     optionsfeasible=(color=green), optionsexcluded=(color=white));
display(domain);
```

(c) By the weak duality theorem, the value of any dual solution must be greater than the value of the primal. Since the primal is unbounded, if the dual had a feasible solution, this would violate weak duality.

End Solution

8. (Duality)

(a) Use the duality definition for the *standard inequality form* to find the dual of the following:

$$\begin{aligned} \max \quad & 4x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & 2x_1 + x_2 + x_3 \leq -1 \\ & x_1 + x_2 + 2x_3 \leq 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(b) Transform the resulting program into a *maximization* problem and again find the dual.

(c) What do you notice?

Solution to task

(a) We introduce dual variables y_1 and y_2 corresponding to the constraints and arrive at the following dual LP:

$$\begin{aligned} \min \quad & -y_1 + 2y_2 \\ \text{s.t.} \quad & 2y_1 + y_2 \geq 4 \\ & y_1 + y_2 \geq 1 \\ & y_1 + 2y_2 \geq 3 \\ & y_1, y_2 \geq 0 \end{aligned}$$

(b) Transform the resulting program into a *maximization* problem and again find the dual. We negate the objective. For convenience we will negate the constraints as well, flipping the direction of the inequality:

$$\begin{aligned} \max \quad & y_1 - 2y_2 \\ \text{s.t.} \quad & -2y_1 - y_2 \leq -4 \\ & -y_1 - y_2 \leq -1 \\ & -y_1 - 2y_2 \leq -3 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Let x_1, x_2, and x_3 be the variables of its dual. Since the problem is in standard inequality form, we can find its dual similarly:

$$\begin{aligned} \min \quad & -4x_1 - x_2 - 3x_3 \\ \text{s.t.} \quad & -2x_1 - x_2 - x_3 \geq 1 \\ & -x_1 - x_2 - 2x_3 \geq -2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

But converting the objective to a maximization problem and negating all constraints, we see that we have our original primal LP:

$$\begin{aligned} \max \quad & 4x_1 + x_2 + 3x_3 \\ \text{s.t.} \quad & 2x_1 + x_2 + x_3 \leq -1 \\ & x_1 + x_2 + 2x_3 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

(c) The dual of the dual of a primal LP is exactly the same primal LP. (And by extension, so is the dual of the dual of the dual of the dual of the primal LP....)

End Solution

9. (Sensitivity with AMPL)
[The AMPL file is posted.]

Zales Jewelers uses rubies and sapphires to produce two types of rings. A Type 1 ring requires 2 rubies, 4 sapphires, and 1 hour of labor. A Type 2 ring requires 4 rubies, 2 sapphires, and 2 hours of labor. Each Type 1 ring sells for \$400; type 2 sells for \$500. At present, Zales has 100 rubies, 120 sapphires, and 70 hours of labor. Extra rubies can be purchased at a cost of \$140 per ruby.

Market demand requires that the company produce at least 15 Type 1 rings and at least 16 Type 2. Let x_1 = number of Type 1 rings produced, x_2 = number of Type 2 rings produced, and r denote the number of rubies purchased. To maximize revenue, Zales solves the following LP:

$$\begin{aligned} \max \quad & 400x_1 + 500x_2 - 140r \\ \text{s.t.} \quad & 2x_1 + 4x_2 - r \leq 100 && (1) \\ & 4x_1 + 2x_2 \leq 120 && (2) \\ & x_1 + 2x_2 \leq 70 && (3) \\ & x_1 \geq 15 && (4) \\ & x_2 \geq 16 && (5) \\ & x_1, x_2, r \geq 0 \end{aligned}$$

An LP has been formulated in AMPL and this is the solution, with corresponding objective value:

$$x_1 = 22; \quad x_2 = 16; \quad r = 8; \quad z = 15,680.$$

The following sensitivity information is generated:

```
: _varname _var.rc _var.down _var.current _var.up    :=
1   x1          0        280          400      1e+20
2   x2          0     -1e+20          500        620
3   r           0       -200         -140       -100 ;

:    _conname   _con.dual    _con.down  _con.current    _con.up      :=
1   constraint1     140     -1e+20           100         108
2   constraint2      30        104           120         184
3   constraint3       0         54            70       1e+20
4   constraint4       0     -1e+20            15          22
5   constraint5    -120          13.3333      16            26.6667 ;
```

Answer the following questions:

(a) Suppose that instead of \$100, each ruby costs \$190. Would Zales still purchase rubies? What would be the new optimal solution to the problem?

(b) Suppose that Zales were only required to produce at least 14 Type 2 rings. What would Zales' revenue be?

(c) What is the most that Zales would be willing to pay for another hour of labor?

(d) What is the most that Zales would be willing to pay for another sapphire?

(e) Zales is considering producing Type 3 rings. Each Type 3 ring can be sold for \$550 and requires 4 rubies, 2 sapphires and 1 hour of labor. Should Zales produce any Type 3 rings? (Hint: you will need to do a small amount of hand calculation here)

Solution to task

(a) The basis remains optimal for ruby objective coefficient between -200 and -100. Thus, a price of \$190 is within the range. The basic feasible solution is unchanged. The revenue decreases by $-50 \cdot 8 = -400$.

(b) From the sensitivity analysis we can see (constraint 5) that decreasing the required number of Type 2 rings by 2 is within the allowable range of change. Thus, the optimal basis does not change.

To get at the change in solution we'd need to look at the effect on $\bar{b}$. But note that the dual doesn't change (since $y^T = c_B^T A_B^{-1}$). Thus, we can get the change in objective from $z = y^T b$. In particular, from the AMPL output we have dual value -120 on this constraint. The proft increases by $\Delta_z = y_5 \cdot \Delta_{b_5} = -120 \cdot -2 = +240$.

(c) Looking at the sensitivity analysis for constraint 3, we see that Zales is not interested in purchasing additional labor because the dual value is zero and thus using $z = y^T b$ the revenue would not change.

(d) Looking at the sensitivty analysis for constraint 2, we see that the corresponding dual value is 30. Because of this, Zales would pay up to $30 for another sapphire.

(e) We can "price out" the new activity (call it x_3). This means computing the reduced cost at the current dual solution. (Note: y doesn't change.)

$$\bar{c_3} = y^T A_3 - c_3 = (1403000 - 120) \cdot \begin{pmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} - 550 = 70$$

Thus, the current basis remains optimal because $\bar{c_3} \geq 0$, and thus Zales should not produce any type 3 rings.

End Solution

10. (Degeneracy)

(a) Describe the condition that makes a feasible tableau degenerate.

(b) Consider the following tableau for a maximization problem. Write down the current basis, current basic solution and current objective value.

$$\begin{array}{cccccccc} z & +\frac{1}{2}x_1 & -2x_2 & & +\frac{3}{2}x_4 & & = & 3 \\ & +\frac{1}{2}x_1 & & +x_3 & +\frac{1}{2}x_4 & & = & 1 \\ & -x_1 & +x_2 & & -x_4 & +x_5 & = & 0 \end{array}$$

(c) Perform two pivots. After each pivot write down the new basis, the new basic solution, and the new objective value. When choosing an entering variable, select the one with the smallest reduced cost. What do you notice?

(d) What is the general problem that can be caused by degeneracy in the simplex method and what is a solution?

Solution to task

(a) A tableau is degenerate if there exists some element in the RHS column of the tableau that is 0.

(b) The current basis is $\{3,5\}$, with $x = (0,0,1,0,0)$ and $z = 3$.

(c) We begin with the following tableau:

$$\begin{bmatrix} var & z & x_1 & x_2 & x_3 & x_4 & x_5 & RHS \\ z & 1 & 0.5 & -2 & 0 & 1.5 & 0 & 3 \\ x_3 & 0 & 0.5 & 0 & 1 & 0.5 & 0 & 1 \\ x_5 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \end{bmatrix}$$

Since only x_2 has a negative coefficient in the reduced cost, it must enter. Since only the coefficient with respect to x_5 has a positive coefficient in that column, x_5 must leave the basis. We perform the pivot:

$$\begin{bmatrix} var & z & x_1 & x_2 & x_3 & x_4 & x_5 & RHS \\ z & 1 & -1.5 & 0 & 0 & -0.5 & 2 & 3 \\ x_3 & 0 & 0.5 & 0 & 1 & 0.5 & 0 & 1 \\ x_2 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \end{bmatrix}$$

Here $x = (0,0,1,0,0)$ and $z = 3$. We let x_1 enter since it has the smallest reduced cost. x_3 has a positive coefficient in that column and is the only candidate for leaving. We perform the pivot:

$$\begin{bmatrix} var & z & x_1 & x_2 & x_3 & x_4 & x_5 & RHS \\ z & 1 & 0 & 0 & 3 & 1 & 2 & 6 \\ x_1 & 0 & 1 & 0 & 2 & 1 & 0 & 2 \\ x_2 & 0 & 0 & 1 & 2 & 0 & 1 & 2 \end{bmatrix}$$

Here $x = (2,2,0,0,0)$ and $z = 6$. We notice

(1) all reduced costs are non-negative, and thus the solution is optimal.

(2) in our first pivot we arrived at the same solution despite performing a pivot.

We used the following Maple commands:

```
with(linalg);
A := <<1,0,0>|<0.5,0.5,-1>|<-2,0,1>|<0,1,0>|<1.5,0.5,-1>|<0,0,1>|<3,1,0>>;
SideVars := vector(3, [z, x_3, x_5]);
Vars := vector(8, [var, z, x_1, x_2, x_3, x_4, x_5, RHS]);
Disp := concat(SideVars,A);
Disp := stackmatrix(Vars,Disp);

A := pivot(A, 3,3);
SideVars := vector(3, [z, x_3, x_2]);
Disp := concat(SideVars,A);
Disp := stackmatrix(Vars,Disp);
```

```
A := pivot(A, 2,2);
A := mulrow(A, 2, 1/A[2,2]);
SideVars := vector(3, [z, x_1, x_2]);
Disp := concat(SideVars,A);
Disp := stackmatrix(Vars,Disp);
```

(d) The simplex method may cycle (returning back to the same tableau). A fix is to use the smallest subscript rule, which guarantees we will not cycle. Rather, we terminate in a finite number of iterations regardless of degeneracy.

End Solution

11. (Pivoting)

Consider the following tableau for a maximization problem:

$$
\begin{array}{cccccccc}
z & & -2x_2 & +2x_3 & & -3x_5 & & = & 5 \\
 & & +2x_2 & -x_3 & & -x_5 & +x_6 & = & 4 \\
 & & +x_2 & -x_3 & +x_4 & +x_5 & & = & 2 \\
 & +x_1 & -2x_2 & -2x_3 & & -3x_5 & & = & 6
\end{array}
$$

(a) List all pairs (x_r, x_k) such that x_k could be the entering variable and x_r could be the leaving variable.

(b) List all pairs if the "most negative reduced cost" rule for choosing the entering variable is used.

(c) List all pairs if the smallest subscript rule is used for choosing the entering and leaving variables.

Solution to task

(a) Note first that the current basis is $\{6, 4, 1\}$. Both x_2 and x_5 may enter (negative reduced cost). If x_2 enters, then either x_6 and x_4 may leave (same ratio of 2). If x_5 enters, only x_4 may leave. Thus, the pairs are (x_6, x_2), (x_4, x_2), and (x_4, x_5).

(b) Now only x_5 may enter (most negative reduced cost). We have (x_4, x_5).

(c) By smallest subscript in entering index, x_2 will enter. By smallest subscript in leaving index, x_4 will leave. We have (x_4, x_2).

End Solution

12. (Sensitivity)

Consider the following LP and its optimal tableau (where x_4 and x_5 are the slack variables for the two inequalities):

$$
\begin{aligned}
\max \quad & 3x_1 + x_2 - x_3 \\
\text{s.t.} \quad & 2x_1 + x_2 + x_3 \leq 8 \\
& 4x_1 + x_2 - x_3 \leq 10 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

$$
\begin{array}{ccccccccc}
z & & & +x_3 & +\frac{1}{2}x_4 & +\frac{1}{2}x_5 & = & 9 \\
& +x_1 & & -x_3 & -\frac{1}{2}x_4 & +\frac{1}{2}x_5 & = & 1 \\
& & +x_2 & +3x_3 & +2x_4 & -x_5 & = & 6
\end{array}
$$

Recall the following algebra relating an original problem in standard equality form and the tableau for basis B:

$$
\begin{aligned}
\bar{b} &= A_B^{-1} b \\
y^T &= c_B^T A_B^{-1} \\
\bar{c}_j &= c_B^T A_B^{-1} A_j - c_j = y^T A_j - c_j, \quad \forall j \in B'
\end{aligned}
$$

Here, the optimal basis is $B = \{1, 2\}$ and

$$
A_B = \begin{pmatrix} 2 & 1 \\ 4 & 1 \end{pmatrix} \quad A_B^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 2 & -1 \end{pmatrix}
$$

Using the optimal tableau or these equations, answer the following questions:

(a) What is the optimal solution to the dual of this LP?

(b) Find the range of values of b_2 (the RHS on the second constraint) for which the current basis remains optimal. If $b_2 = 12$ what is the new optimal solution?

(c) Find the range of values of c_3 (the objective function coefficient on the third variable) for which the current solution remains optimal.

(d) Find the range of values of c_1 (the objective function coefficient on the first variable) for which the current basis remains optimal. (**Note: it is more important you understand how to solve this problem than to actually solve it here. It's good algebra practice, so we won't stop you, but don't expect us to give this much algebra on the midterm itself.**)

Solution to task

(a) Either, read off the optimal reduced cost of the variables corresponding to the isolated, slack variables from the initial formulation (x_4 and x_5).

Or, use

$$y^T = c_B^T A_B^{-1} = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} .5 & .5 \end{bmatrix}$$

[Note: one way to check your answer is to confirm that the objective value of the dual $\bar{b}^T y = 8y_1 + 10y_2 = 8(.5) + 10(.5) = 9$ equals that of the optimal primal.]

(b) Because $\bar{c}$ does not change with a change in b, we can concentrate on $\bar{b}$. In particular we need $\bar{b} = A_B^{-1} b \geq 0$ to retain feasibility. Consider $b_2 := 10 + \epsilon$. This gives us:

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 8 \\ 10+\epsilon \end{bmatrix} =$$
$$\begin{bmatrix} -\frac{1}{2}(8) + \frac{1}{2}(10+\epsilon) \\ 2(8) - 1(10+\epsilon) \end{bmatrix} =$$
$$\begin{bmatrix} 1 + \frac{\epsilon}{2} \\ 6 - \epsilon \end{bmatrix} \geq \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Which gives us:

$$1 + \frac{\epsilon}{2} \geq 0$$
$$6 - \epsilon \geq 0$$

Or:

$$6 \geq \epsilon \geq -2$$

So $16 \geq b_2 \geq 8$ will ensure that the current optimal solution remains valid.

For $b_2 := 12$, we can read off the revised, optimal solution as $\bar{b}$, and thus $x_1 = 1 + \epsilon/2 = 1 + 2/2 = 2$ and $x_2 = 6 - \epsilon = 6 - 2 = 4$.

(c) Since $\bar{c}_3$ is a nonbasic variable in the final tableau, we need to make sure that $\bar{c}_3$ remains non-negative. All other $\bar{c}_j, j \neq 3$ and $\bar{b}$ will remain the same. Consider $c_3 := -1 + \epsilon$. We need:

$$\bar{c}_3 = y^T A_3 - (c_3 + \epsilon) \geq 0$$

(Note that y^T does not change when $\bar{c}_3$ changes by $y^T = c_B^T A_B^{-1}$.) Substituting we have:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - (-1 + \epsilon) \geq 0$$

Or:

$$\epsilon \leq 1$$

Thus, we need $c_3 \leq 0$ for the current solution to remain optimal.

Alternate solution: Because x_3 is a nonbasic variable you can get the same information by noting that the reduced cost $\overline{c}_3 = 1$ in the optimal tableau, which shows that the objective function coefficient on x_3 would have to increase by more than 1 for the current basis not to be optimal. Turning to the original problem, we have $c_3 = -1$, so $c_3 \leq 0$ gives the range of values of c_3 for which the current solution remains optimal.

(d) We need to check that the reduced cost on every nonbasic variable remains non-negative. First we determine the effect of changing $c_1 := 3 + \epsilon$ on the dual solution y:

$$y^T = c_B^T A_B^{-1}$$

Substituting:

$$y^T = \begin{bmatrix}(3+\epsilon) & 1\end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 2 & -1 \end{bmatrix} =$$
$$\begin{bmatrix}(3+\epsilon)(-\frac{1}{2}) + 1(2) & (3+\epsilon)(\frac{1}{2}) + 1(-1)\end{bmatrix} =$$
$$\begin{bmatrix}-\frac{3}{2} - \frac{\epsilon}{2} + 2 & \frac{3}{2} + \frac{\epsilon}{2} - 1\end{bmatrix} = \begin{bmatrix}\frac{1}{2} - \frac{\epsilon}{2} & \frac{1}{2} + \frac{\epsilon}{2}\end{bmatrix}$$

Now we check conditions for all the non-basic variable coefficients $\bar{c_3}, \bar{c_4}, \bar{c_5}$ to remain non-negative in the final tableau:

c_3:

$$\overline{c}_3 = y^T A_3 - c_3 \geq 0$$

Substituting we have:

$$\begin{bmatrix}\frac{1}{2} - \frac{\epsilon}{2} & \frac{1}{2} + \frac{\epsilon}{2}\end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - (-1) \geq 0$$

Or:

$$(\frac{1}{2} - \frac{\epsilon}{2} - \frac{1}{2} - \frac{\epsilon}{2}) - (-1) \geq 0$$

Or:

$$-\epsilon + 1 \geq 0$$

Or:

$$\epsilon \leq 1$$

c_4:

$$\overline{c}_4 = y^T A_4 - c_4 \geq 0$$

Substituting we have:

$$\begin{bmatrix}\frac{1}{2} - \frac{\epsilon}{2} & \frac{1}{2} + \frac{\epsilon}{2}\end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 0 \geq 0$$

Or:

$$\frac{1}{2} + \frac{\epsilon}{2} - 0 \geq 0$$

Or:

$$\epsilon \leq 1$$

c_5:

$$\bar{c}_5 = y^T A_5 - c_5 \geq 0$$

Substituting we have:

$$\begin{bmatrix} \frac{1}{2} - \frac{\epsilon}{2} & \frac{1}{2} + \frac{\epsilon}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 0 \geq 0$$

Or:

$$\frac{1}{2} - \frac{\epsilon}{2} - 0 \geq 0$$

Or:

$$\epsilon \geq -1$$

Putting these together, the largest range of ϵ for which all are satisfied is $-1 \leq \epsilon \leq 1$, which is equivalent to $2 \leq c_1 \leq 4$.

As a sanity check we again use AMPL:

```
var x1 >= 0;
var x2 >= 0;
var x3 >= 0;

maximize objective: 3*x1 + x2 - x3;

subject to constraint1: 2*x1 + x2 + x3 <= 8;
subject to constraint2: 4*x1 + x2 - x3 <= 10;
--------------------
reset;
reset data;

model matrixsens.mod;

option cplex_options 'sensitivity primalopt presolve 0';
option presolve 0;
solve;

display x1,x2,x3;

display _varname, _var.rc, _var.down, _var.current, _var.up > matrixsens.sens;
display _conname, _con.dual, _con.down, _con.current, _con.up > matrixsens.sens;
--------------------
: _varname _var.rc   _var.down _var.current _var.up    :=
1   x1         0          2           3        4
2   x2         0          0.75        1        1.5
3   x3        -1     -1e+20          -1        0
;
```

```
:    _conname   _con.dual _con.down _con.current _con.up    :=
1   constraint1     0.5        5            8         10
2   constraint2     0.5        8           10         16
;
```

And we see that all our algebra was correct. BAM! **End Solution**

13. (Dual simplex)

In solving the following LP

$$\begin{aligned} \max \quad & 6x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 5 \\ & 2x_1 + x_2 \le 6 \\ & x_1, x_2 \ge 0 \end{aligned}$$

we obtain the optimal tableau

$$\begin{array}{rrrrcl} z & +2x_2 & & +3x_4 & = & 18 \\ & +\frac{1}{2}x_2 & +x_3 & -\frac{1}{2}x_4 & = & 2 \\ +x_1 & +\frac{1}{2}x_2 & & +\frac{1}{2}x_4 & = & 3 \end{array}$$

where x_3 and x_4 are the slack variables introduced for the two inequalities.

(a) Find the optimal solution if we add the constraint $3x_1 + x_2 \le 10$ to the original LP.

(b) Find the optimal solution if we add the constraint $x_1 - x_2 \ge 6$ to the original LP.

(c) Find the optimal solution if we add the constraint $8x_1 + x_2 \le 12$ to the original LP.

Solution to task

1. Because all reduced costs in the current tableau are non-negative, we are at an optimal solution. We can read off the current solution from the tableau, it is $x = (3, 0, 2, 0)$. Thus, it is easy to see that after adding the new constraint, the current solution is still feasible. Thus, the optimal solution doesn't change.

2. The new constraint makes the current basis infeasible:

$$\begin{array}{rrrrrcl} z & +2x_2 & & +3x_4 & & = & 18 \\ & +\frac{1}{2}x_2 & +x_3 & -\frac{1}{2}x_4 & & = & 2 \\ +x_1 & +\frac{1}{2}x_2 & & +\frac{1}{2}x_4 & & = & 3 \\ -x_1 & +x_2 & & & +x_5 & = & -6 \end{array}$$

We add row 3 to row 4 to isolate the basic variables. We get the following tableau:

$$\begin{array}{lllllll} z & & +2x_2 & & +3x_4 & = & 18 \\ & & +\frac{1}{2}x_2 & +x_3 & -\frac{1}{2}x_4 & = & 2 \\ & +x_1 & +\frac{1}{2}x_2 & & +\frac{1}{2}x_4 & = & 3 \\ & & +1\frac{1}{2}x_2 & & +\frac{1}{2}x_4 & +x_5 = & -3 \end{array}$$

This tableau is not primal feasible, but it is dual feasible. We can apply the dual simplex. We pivot on the last row to have x_5 exit. However, all $\bar{a}_{rj}$ from $j \notin B$ are nonnegative and thus the dual is unbounded, and the primal is infeasible (by weak duality).

3. Adding this constraint we get the following tableau:

$$\begin{array}{lllllll} z & & +2x_2 & & +3x_4 & = & 18 \\ & & +\frac{1}{2}x_2 & +x_3 & -\frac{1}{2}x_4 & = & 2 \\ & +x_1 & +\frac{1}{2}x_2 & & +\frac{1}{2}x_4 & = & 3 \\ & 8x_1 & +x_2 & & & +x_5 = & 12 \end{array}$$

To isolate the basic variable, we subtract 8 times row 3 from row 4 and we get:

$$\begin{array}{lllllll} z & & +2x_2 & & +3x_4 & = & 18 \\ & & +\frac{1}{2}x_2 & +x_3 & -\frac{1}{2}x_4 & = & 2 \\ & +x_1 & +\frac{1}{2}x_2 & & +\frac{1}{2}x_4 & = & 3 \\ & & -3x_2 & & -4x_4 & +x_5 = & -12 \end{array}$$

This tableau is dual feasible. We can apply dual simplex. We pivot on the last row to have x_5 exit. The minimum ratio test $(-2/-3$ vs. $-3/-4)$ tells us to bring x_2 into the basis. Doing so leads to the following tableau:

$$
\begin{array}{llllllll}
z & & & & +\frac{1}{3}x_4 & +\frac{2}{3}x_5 = & 10 \\
 & & & +x_3 & -\frac{7}{6}x_4 & -\frac{1}{6}x_5 = & 0 \\
 & +x_1 & & & -\frac{1}{6}x_4 & -\frac{1}{6}x_5 = & 1 \\
 & & +x_2 & & \frac{4}{3}x_4 & -\frac{1}{3}x_5 = & 4
\end{array}
$$

And we are done! The RHS is non-negative, thus we have a primal feasible solution and the solution is optimal because all reduced costs are non-negative. Thus, the new solution is $x = (1, 4, 0, 0)$ and the new objective value is $z = 10$.

End Solution