Lecture 2: Lesson Plan

- What is an LP?
- Graphical and algebraic correspondence
- Problems in canonical form
- LP in matrix form. Matrix review.

Jensen & Bard: 2.1-2.3, 2.5, 3.1 (can ignore the two definitions for now), 3.2

Available in Gordon McKay library (3rd floor of Piece Hall).
Linear Programming

- Maximizing (or minimizing) a linear function subject to a finite number of linear constraints

\[
\max \sum_{j=1}^{n} c_j x_j
\]

subject to

\[
\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, \ldots, m)
\]

\[
x_j \geq 0 \quad (j = 1, \ldots, n)
\]

Decision variables: \( x_j \)
Parameters: \( c_j, a_{ij} \)

Standard Inequality Form

\[
\max \quad c^T x
\]

subject to

\[
A x \leq b
\]

\[
x \geq 0
\]

\[
c^T = (c_1, \ldots, c_n)
\]

\[
A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}
\]
标准等价形式

\[
\begin{align*}
\text{max} & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

\[c^T = (c_1, \ldots, c_n)\]

\[A = \begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{m1} & \cdots & a_{mn}
\end{pmatrix}, \quad x = \begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix}, \quad b = \begin{pmatrix}
b_1 \\
\vdots \\
b_m
\end{pmatrix}
\]

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Terminology for Solutions of LP

• A feasible solution
  – A solution that satisfies all constraints

• An infeasible solution
  – A solution that violates at least one constraint

• Feasible region
  – The region of all feasible solutions

• An optimal solution
  – A feasible solution that has the most favorable value of the objective function

Example: Marketing Campaign

• Ad on news page– get 7m high-income women, 2m high-income men. $50,000

• Ad on sports page– get 2m high-income women and 12m high-income men. $100,000

• Goal: 28m women, 24m men; min cost. How many of each ad to buy? (Can buy fractions!)

\[
\begin{align*}
\text{min } & z = 50x_1 + 100x_2 \\
\text{s.t. } & 7x_1 + 2x_2 \geq 28 \\
& 2x_1 + 12x_2 \geq 24 \\
& x_1, x_2 \geq 0
\end{align*}
\]
Graphical version of problem
(solution is \( x_1=3.6, \ x_2=1.4, \) value 320)

\[
\begin{align*}
\text{min} \quad & z = 50x_1 + 100x_2 \\
\text{s.t.} \quad & 7x_1 + 2x_2 \geq 28 \\
& 2x_1 + 12x_2 \geq 24 \\
& x_1, x_2 \geq 0
\end{align*}
\]

Solution is at an extreme point of feasible region!

Example: Multiple Opt. Solutions

\[
\begin{align*}
\text{max} \quad & 3x_1 - x_2 \\
\text{s.t.} \quad & 15x_1 - 5x_2 \leq 30 \\
& 10x_1 + 30x_2 \leq 120 \\
& x_1, x_2 \geq 0
\end{align*}
\]

Note: still extremal optimal solutions
Example: Unbounded Objective

\[
\begin{align*}
\text{max} & \quad -x_1 + x_2 \\
\text{s.t.} & \quad -x_1 + 4x_2 \geq 0 \\
& \quad x_1 \leq 4 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]
Solving LPs

- Transform to the **canonical form** (note: this is NOT the “standard equality form”)
- Work with **basic feasible solutions**
- **Iterate**: solution improvement
  - From one BFS to the next...

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Canonical Form

\[
\begin{align*}
\text{max } z &= 0x_1 + 0x_2 - 3x_3 - x_4 + 20 \\
\text{s.t.} &
\begin{align*}
x_1 &- 3x_3 + 3x_4 = 6 \\
x_2 &- 8x_3 + 4x_4 = 4 \\
x_j &\geq 0
\end{align*}
\end{align*}
\]

1. **Maximization**
2. **RHS coefficients are non-negative**
3. All constraints are **equalities**
4. **Decision variables all non-negative**
5. One decision variable is “**isolated**” in each constraint:
   - a +1 coefficient.
   - does not appear in any other constraint
   - zero coefficient in objective

Why might this be useful??
Basic Feasible Solution

\[
\begin{align*}
\text{max } z &= 0x_1 + 0x_2 - 3x_3 - x_4 + 20 \\
\text{s.t. } & \quad -3x_3 + 3x_4 = 6 \\
& \quad -8x_3 + 4x_4 = 4 \\
& \quad x_j \geq 0
\end{align*}
\]

Canonical form has an associated basic feasible solution in which the isolated variables (basic vars) are non-zero and the rest (non-basic vars) are zero.

Here, set \(x_1 = 6, x_2 = 4, x_3 = 0, x_4 = 0\).
Basic Feasible Solution

\[
\begin{align*}
\max z &= 0x_1 + 0x_2 - 3x_3 - x_4 + 20 \\
\text{s.t.} & \quad x_1 \\
& \quad -3x_3 + 3x_4 = 6 \\
& \quad -8x_3 + 4x_4 = 4 \\
& \quad x_j \geq 0
\end{align*}
\]

Canonical form has an associated basic feasible solution in which the isolated variables (basic vars) are non-zero and the rest (non-basic vars) are zero.

Here, set \(x_1 = 6, x_2 = 4, x_3 = 0, x_4 = 0\).

Optimal in this example as well. (Why?)

Solution Improvement

\[
\begin{align*}
\max z &= 0x_1 + 0x_2 - 3x_3 + x_4 + 20 \\
\text{s.t.} & \quad x_1 \\
& \quad -3x_3 + 3x_4 = 6 \\
& \quad -8x_3 + 4x_4 = 4 \\
& \quad x_j \geq 0
\end{align*}
\]

Current BFS: \(x_1 = 6, x_2 = 4, x_3 = 0, x_4 = 0\).
Solution Improvement

\[
\begin{align*}
\text{max } z &= 0x_1 + 0x_2 - 3x_3 + x_4 + 20 \\
\text{s.t.} \\
x_1 &= -3x_3 + 3x_4 = 6 \\
x_2 &= -8x_3 + 4x_4 = 4 \\
x_j &\geq 0
\end{align*}
\]

Current BFS: \( x_1 = 6, x_2 = 4, x_3 = 0, x_4 = 0 \).
Let’s increase \( x_4 \). Need to decrease \( x_1 \) and \( x_2 \) (keep \( x_3 = 0 \)) to keep feasible.

Solution Improvement

\[
\begin{align*}
\text{max } z &= 0x_1 + 0x_2 - 3x_3 + x_4 + 20 \\
\text{s.t.} \\
x_1 &= -3x_3 + 3x_4 = 6 \\
x_2 &= -8x_3 + 4x_4 = 4 \\
x_j &\geq 0
\end{align*}
\]

Current BFS: \( x_1 = 6, x_2 = 4, x_3 = 0, x_4 = 0 \).
Let’s increase \( x_4 \). Need to decrease \( x_1 \) and \( x_2 \) (keep \( x_3 = 0 \)) to keep feasible. Second constraint becomes binding.
Obtain new solution: \( x_1 = 3, x_2 = 0, x_3 = 0, x_4 = 1 \). Value 21.
Solution Improvement

$$\begin{align*}
\text{max } z &= 0x_1 + 0x_2 - 3x_3 + x_4 + 20 \\
\text{s.t.} & \quad x_1 - 3x_3 + 3x_4 = 6 \\
& \quad x_2 - 8x_3 + 4x_4 = 4 \\
& \quad x_j \geq 0
\end{align*}$$

Current BFS: $x_1 = 6, x_2 = 4, x_3 = 0, x_4 = 0$.
Let’s increase $x_4$. Need to decrease $x_1$ and $x_2$ (keep $x_3 = 0$) to keep feasible. Second constraint becomes binding.
Obtain new solution: $x_1 = 3, x_2 = 0, x_3 = 0, x_4 = 1$. Value 21.

Corresponds to a new canonical form. Isolated vars: $x_1$ and $x_4$.
“pivot on $x_4$ in the second constraint”
“pick something to enter, something forced to leave”

New Canonical Form

After linear transformations:

$$\begin{align*}
\text{max } z &= 0x_1 - \frac{1}{4}x_2 - x_3 + 0x_4 + 21 \\
\text{s.t.} & \quad x_1 - \frac{3}{4}x_2 + 3x_3 = 3 \\
& \quad \frac{1}{4}x_2 - 2x_3 + x_4 = 1 \\
& \quad x_j \geq 0
\end{align*}$$

New BFS is $x_1 = 3, x_2 = 0, x_3 = 0, x_4 = 1$, and optimal.
Geometric Interpretation of Solution Improvement

\begin{align*}
\max z &= 0x_1 + 0x_2 - 3x_3 + x_4 + 20 \\
\text{s.t.} \\
\left\{\begin{array}{c}
x_1 - 3x_3 + 3x_4 = 6 \\
x_2 - 8x_3 + 4x_4 = 4 \\
x_j \geq 0
\end{array}\right.
\end{align*}

\begin{align*}
x_1 &= 3x_3 - 3x_4 + 6 \geq 0 \\
x_2 &= 8x_3 - 4x_4 + 4 \geq 0
\end{align*}

\begin{align*}
x_1=6, x_2=4, x_3=0, x_4=0 \\
\downarrow
\end{align*}

\begin{align*}
x_1=3, x_2=0, x_3=0, x_4=1
\end{align*}

Can any LP be made canonical?

\begin{align*}
\max z &= 0x_1 - \frac{1}{4}x_2 - x_3 + 0x_4 + 21 \\
\text{s.t.} \\
\left\{\begin{array}{c}
x_1 - \frac{3}{4}x_2 + 3x_3 = 3 \\
\frac{1}{4}x_2 - 2x_3 + x_4 = 1 \\
x_j \geq 0
\end{array}\right.
\end{align*}

(1) maximization, (2) positive RHS, (3) equality constraints, (4) non-negative vars, (5) isolated vars.

\emph{+1 coeff, only in one constraint, not in obj.}
Reduction to canonical form (I)

• “min z” =
• If a RHS value is negative then

• If $x_1 \leq 0$ then

• If $x_3$ is “free” (neither $x_3 \leq 0$ or $x_3 \geq 0$) then

Reduction to canonical form (I)

• “min z” = “max –z”
• If a RHS value is negative then

• If $x_1 \leq 0$ then

• If $x_3$ is “free” (neither $x_3 \leq 0$ or $x_3 \geq 0$) then
Reduction to canonical form (I)

• “min z” = “max –z”
• If a RHS value is negative then multiply constraint by -1

• If $x_1 \leq 0$ then

• If $x_3$ is “free” (neither $x_3 \leq 0$ or $x_3 \geq 0$) then

Reduction to canonical form (I)

• “min z” = “max –z”
• If a RHS value is negative then multiply constraint by -1

• If $x_1 \leq 0$ then replace $x_1 := -x_2$, with $x_2 \geq 0$

• If $x_3$ is “free” (neither $x_3 \leq 0$ or $x_3 \geq 0$) then
Reduction to canonical form (I)

- “min z” = “max –z”
- If a RHS value is negative then multiply constraint by -1
- If \( x_1 \leq 0 \) then replace \( x_1 := -x_2 \), with \( x_2 \geq 0 \)
- If \( x_3 \) is “free” (neither \( x_3 \leq 0 \) or \( x_3 \geq 0 \)) then replace \( x_3 := u - v \), with \( u \geq 0 \) and \( v \geq 0 \).

Reduction to canonical form (II)

- Inequality constraints
  
  \[
  40x_1 + 10x_2 + 6x_3 \leq 55 \\
  40x_1 + 10x_2 + 6x_3 \geq 33
  \]
Reduction to canonical form (II)

- Inequality constraints

\[
\begin{align*}
40x_1 + 10x_2 + 6x_3 & \leq 55 \\
40x_1 + 10x_2 + 6x_3 & \geq 33
\end{align*}
\]

\[
\begin{align*}
40x_1 + 10x_2 + 6x_3 + x_4 & = 55 \\
40x_1 + 10x_2 + 6x_3 - x_5 & = 33
\end{align*}
\]

\[x_4 \geq 0, \quad x_5 \geq 0\]

Reduction to canonical form (III)

- Need isolated variables
- A constraint with slack var already good!

\[
\begin{align*}
40x_1 + 10x_2 + 6x_3 + x_4 & = 55
\end{align*}
\]
Reduction to canonical form (III)

• Need isolated variables
• A constraint with slack var already good!

\[ 40x_1 + 10x_2 + 6x_3 + x_4 = 55 \]

• Other constraints, e.g. with surplus vars not good:

\[ 40x_1 + 10x_2 + 6x_3 - x_5 = 33 \]

doesn’t work

• Introduce a new artificial variable (we’ll insist that \( x_6 = 0 \) in any solution)

\[ 40x_1 + 10x_2 + 6x_3 - x_5 + x_6 = 33 \]
Standard Inequality Form

$$\text{max } c^T x$$

s.t.  \(Ax \leq b\)

\(x \geq 0\)

\(c^T = (c_1, \ldots, c_n)\)

\(A = \begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & & \vdots \\
a_{m1} & \cdots & a_{mn}
\end{pmatrix}
\)

\(x = \begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix}
\)

\(b = \begin{pmatrix}
b_1 \\
\vdots \\
b_m
\end{pmatrix}
\)

Review: Matrices (1/4)

- Matrix: rectangular array of numbers \([a_{ij}]\)
  - dimension: \(m \times n\) (\(m\) rows, \(n\) columns)
  - \(k \times 1\): column vector; \(1 \times k\): row vector

- \(B = \alpha A = A\alpha\), scalar \(\alpha\): \(\alpha a_{ij} = b_{ij}\)
Review: Matrices (1/4)

• Matrix: rectangular array of numbers $[a_{ij}]$
  – dimension: $m$ by $n$ ($m$ rows, $n$ columns)
  – $k$ by 1: column vector; 1 by $k$: row vector

• $B = \alpha A = A\alpha$, scalar $\alpha$: $\alpha a_{ij} = b_{ij}$

$$A \cdot B = C$$

\[
\begin{pmatrix}
2 & 6 & -3 \\
1 & 4 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
0 & -3 \\
3 & 1 \\
\end{pmatrix}
= \begin{pmatrix}
-7 & -17 \\
1 & -10 \\
\end{pmatrix}
\]

Review: Matrices (2/4)

• $A^T$ transpose: $a^T_{ij} = a_{ji}$

\[
A = \begin{pmatrix}
2 & 4 & -1 \\
-3 & 0 & 44 \\
\end{pmatrix}
A^T = \begin{pmatrix}
2 & -3 \\
4 & 0 \\
-1 & 4 \\
\end{pmatrix}
\]

• $c^T \cdot x = \sum_{j=1}^{n} c_j x_j$ (1 x n) (n x 1) “inner product”
Review: Matrices (2/4)

- $A^T$ transpose: $a^T_{ij} = a_{ji}$
  
  $A = \begin{pmatrix} 2 & 4 & -1 \\ -3 & 0 & 44 \end{pmatrix}$ \hspace{1cm} $A^T = \begin{pmatrix} 2 & 4 \\ -3 & 0 \\ 44 & -1 \end{pmatrix}$

- $c^T \cdot x = \sum_{j=1}^{n} c_j x_j$ \hspace{1cm} “inner product”

- Partitions
  
  $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} A_1 & \cdots & A_n \end{pmatrix}$

  $Ax = A_1 x_1 + \ldots + A_n x_n$

Review: Matrices (3/4)

- **Square** matrix: $m$ by $m$
- **Identity** matrix: square matrix w/ diagonal elements all 1 and all non-diagonal are 0.

- $I_2, I_3, \ldots$

  $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ \hspace{1cm} $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

- $m$ by $m$ square $A$, inverse: $A^{-1} = B \Rightarrow BA = AB = I_m$

  $\begin{pmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -5 & 1 & -7 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Review: Matrices (4/4)

• Given $Ax = b$ (with square matrix $A$)
• Can write:
  \[ A^{-1}(Ax) = A^{-1}b \]

  Equivalently:
  \[ x = A^{-1}b \]

• Can find a unique solution to a square linear system if $A$ is invertible.

Next Time

• Applications, Examples, Exercises.