Selected applications of linear optimization

AM121

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Cancer and Radiation Therapy

• **14.5 million** Americans with a history of cancer were alive on Jan 1, 2014.

• Estimated **1.4 million** new cancer cases diagnosed in 2015.

• Around **two-thirds** of cancer patients will receive radiation therapy during their illness.

• X-rays, gamma-rays, charged particles

• Use of radiation to treat cancer is about 100 years old!
Conventional Radiation Therapy: Uniform Beam Strength
Challenge: Need to protect regions

Dashed line: 90% of target dose
Intensity-Modulated Radiation Therapy (IMRT): Non-uniform beam strengths to match target better

Control of beam shape with Multi-Leaf Collimator (MLC)
Decompose single beam into many “beamlets”
Related Problem:
How to draw a picture with only straight lines?

Birkhoff GD (1940) On drawings composed of uniform straight lines
Simple Model

• Discretize target area into “voxels”

• Each beamlet has a fixed amount of radiation over each voxels

• **Objective:** choose weights of beams to meet target levels of radiation dose in certain areas
  • High radiation in tumor cells
  • Low radiation in normal cells
  • Very low radiation in critical structures
Result of simple LP formulation

Unweighted objective

Weighted objective
With cutoff on maximum beam

Max weight $\leq 5 \times$ average weight
Applications of Robust Optimization
SailCo (Lecture 3) simplified, revisited

- SailCo must decide how many sailboats to produce each quarter. Must meet demand in each Q. Boats made in a Q can be used to meet demand in same Q.

<table>
<thead>
<tr>
<th></th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forecasted demand</td>
<td>40</td>
<td>60</td>
<td>75</td>
<td>25</td>
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</tbody>
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- Production cost: $400 / boat (simpler!)

- Inventory cost: $20/boat/quarter for boats on hand at the end of the quarter

- 10 boats at start of Q1.

- **Formulate an LP to determine a production schedule to minimize the sum of production and inventory cost**
SailCo formulation

\[
\min z = 400 \sum_{t} x_t + 20 \sum_{t} h_t
\]

s.t. \quad h_1 = 10 + x_1 - 40

\quad h_2 = h_1 + x_2 - 60

\quad h_3 = h_2 + x_3 - 75

\quad h_4 = h_3 + x_4 - 25

\quad x_t, h_t \geq 0

x_t : Number of boats produced in Quarter t

h_t : Inventory of boats at end of Quarter t
What if demand is uncertain?

• Suppose demand is not known exactly

• In a quarter, produce boats before observing demand

• Unmet demand is backlogged, but incurs penalty cost of $1,000 / boat / Q

• End-of-Quarter inventory can be negative (backlog)
Formulation with uncertain demand

\[
\min z = 400 \sum_t x_t + 20 \sum_t h_t^+ + 1000 \sum_t h_t^-
\]

s.t. \[ h_1 = 10 + x_1 - \tilde{d}_1, \quad \tilde{d}_1 \in [35, 45] \]
\[ h_2 = h_1 + x_2 - \tilde{d}_2, \quad \tilde{d}_2 \in [50, 70] \]
\[ h_3 = h_2 + x_3 - \tilde{d}_3, \quad \tilde{d}_3 \in [60, 90] \]
\[ h_4 = h_3 + x_4 - \tilde{d}_4, \quad \tilde{d}_4 \in [20, 30] \]
\[ x_t \geq 0 \]

(Positive Part) \[ A^+ := \max \{0, A\} \]
(Negative Part) \[ A^- := \max \{0, -A\} \]
Formulation with uncertain demand (linear)

\[ \min z = 400 \sum_t x_t + 20 \sum_t a_t + 1000 \sum_t b_t \]

s.t. \[ h_1 = 10 + x_1 - \tilde{d}_1, \quad \tilde{d}_1 \in [35, 45] \]
\[ h_2 = h_1 + x_2 - \tilde{d}_2, \quad \tilde{d}_2 \in [50, 70] \]
\[ h_3 = h_2 + x_3 - \tilde{d}_3, \quad \tilde{d}_3 \in [60, 90] \]
\[ h_4 = h_3 + x_4 - \tilde{d}_4, \quad \tilde{d}_4 \in [20, 30] \]
\[ h_t = a_t - b_t \quad t = 1, 2, 3, 4 \]
\[ x_t, a_t, b_t \geq 0 \]
Adapted Decisions (aka Decision Rules)

\[ x_1 \]

\[ x_2(\tilde{d}_1) \]

\[ x_3(\tilde{d}_1, \tilde{d}_2) \]

\[ x_4(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3) \]

\[
\begin{align*}
\text{s.t.} & \quad h_1 = 10 + x_1 - d_1, \quad d_1 \in [35, 45] \\
& \quad h_2 = h_1 + x_2 - \tilde{d}_2, \quad \tilde{d}_2 \in [50, 70] \\
& \quad h_3 = h_2 + x_3 - \tilde{d}_3, \quad \tilde{d}_3 \in [60, 90] \\
& \quad h_4 = h_3 + x_4 - \tilde{d}_4, \quad \tilde{d}_4 \in [20, 30] \\
& \quad h_t = a_t - b_t \quad t = 1, 2, 3, 4 \\
& \quad x_t, a_t, b_t \geq 0 
\end{align*}
\]

\[
\begin{align*}
h_1(\tilde{d}_1) & \quad := \quad h_1^0 + h_1^1\tilde{d}_1 \\
h_2(\tilde{d}_1, \tilde{d}_2) & \quad := \quad h_2^0 + h_2^1\tilde{d}_1 + h_2^2\tilde{d}_2 \\
h_3(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3) & \quad := \quad h_3^0 + h_3^1\tilde{d}_1 + h_3^2\tilde{d}_2 + h_3^3\tilde{d}_3 \\
h_4(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4) & \quad := \quad h_4^0 + h_4^1\tilde{d}_1 + h_4^2\tilde{d}_2 + h_4^3\tilde{d}_3 + h_4^4\tilde{d}_4
\end{align*}
\]
Objective: Worst-case

\[
\min_{x_t, h_t, a_t, b_t} \max_{\tilde{d}} \left\{ 400 \sum_t x_t(\tilde{d}) + 20 \sum_t a_t(\tilde{d}) + 1000 \sum_t b_t(\tilde{d}) \right\}
\]

\[
\begin{align*}
x_1 &:= x_1^0 \\
x_2(\tilde{d}_1) &:= x_2^0 + x_2^1 \tilde{d}_1 \\
x_3(\tilde{d}_1, \tilde{d}_2) &:= x_3^0 + x_3^1 \tilde{d}_1 + x_3^2 \tilde{d}_2 \\
x_4(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3) &:= x_4^0 + x_4^1 \tilde{d}_1 + x_4^2 \tilde{d}_2 + x_4^3 \tilde{d}_3
\end{align*}
\]
Project Management

- **Project**: Set of tasks that need to be done in a certain sequence
- E.g., consulting project, construction project, scientific project,
- Project completes when all tasks are complete
- **Problem**: How to efficiently allocate resources to tasks? Minimize completion time? Completion cost?
Activity on Arc (AoA) Project Network
LP to figure out project completion time

- **For now:** Assume task durations known exactly, no resource allocation.

\[
\begin{align*}
\text{min} & \quad x_N \\
\text{s.t.} & \quad x_{j(a)} \geq x_{i(a)} + d_a \quad a \in A \\
& \quad x_n \geq 0
\end{align*}
\]

- \( N \): final node
- \( A \): set of all arcs (tasks)
- \( x_n \): earliest time of node \( n \)
- \( i(a) \): starting node of arc \( a \)
- \( j(a) \): ending node of arc \( a \)
Project Crashing

- Add resources to reduce task length ("crashing")
- Deterministic task times
- Linear cost of crashing
- Target completion time with penalty

\[
\min \ p(x_N - \tau)^+ + \sum_a c_a y_a \\
\text{s.t.} \quad x_{j(a)} \geq x_{i(a)} + d_a - y_a \quad a \in A \\
\quad x_n \geq 0 \\
\quad 0 \leq y_a \leq u_a
\]

\(\tau\): target completion time  
\(p\): penalty on lateness  
\(u_a\): upper limit on crash amount  
\(c_a\): unit cost of crashing task \(a\)  
\(y_a\): amount that task \(a\) is crashed
What if task times are not precisely known?

\[
\begin{align*}
\min & \quad p(x_N - \tau)^+ + \sum_a c_a y_a \\
\text{s.t.} \quad x_j(a) & \geq x_i(a) + \overline{d}_a - y_a \\
& \quad \overline{d}_a \in \mathcal{U}_a, a \in \mathcal{A} \\
& \quad x_n \geq 0 \\
& \quad 0 \leq y_a \leq u_a
\end{align*}
\]

\[\mathcal{U}_a:\text{ uncertainty set for task } a\]
How about using LDRs?

\[
\begin{align*}
\min & \quad p(x_N(\tilde{d}) - \tau)^+ + \sum_a c_a y_a(\tilde{d}) \\
\text{s.t.} & \quad x_{j(a)}(\tilde{d}) \geq x_{i(a)}(\tilde{d}) + \tilde{d}_a - y_a(\tilde{d}) \quad \tilde{d}_a \in \mathcal{U}_a, a \in A \\
& \quad x_n(\tilde{d}) \geq 0 \\
& \quad 0 \leq y_a(\tilde{d}) \leq u_a
\end{align*}
\]

\(x_n(\cdot)\): Adapted LDR for earliest time of node \(n\)

\(y_a(\cdot)\): Adapted LDR for crash amount of task \(a\)
What does adaptedness mean?
Summary

- Linear programming formulations for IMRT treatment planning
- Robust linear optimization and the role of duality
- Controlling the level of conservativism
- Adapted robust optimization in production planning
- Project management
HBS 4465 / AM 222: Stochastic Modeling

The course covers the modeling, analysis, and control of stochastic systems. Topics include Bernoulli and Poisson processes, Markov chains and Markov decision processes, optimization under uncertainty, queuing theory, and simulation. Applications will be presented in healthcare, inventory management, and service systems.