AM 121: Intro to Optimization Models and Methods

EO 2: Paired Kidney Donation

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Matching Problems

• One-sided matching with strict preferences
• One-sided matching
• Two-sided matching (guest lecture on Monday)
One-sided Matching (strict prefs)

- Each agent “owns” an item (dorm rooms?)
- Strict preferences
- Solutions:
  - Random serial dictatorship (RSD)
  - Top trading cycle algorithm

Top trading cycles

2. Select a cycle and trade, agents that trade leave the market.

(Hartline/Immorlica)
One-Sided Matching (0/1 prefs)

Kidney disease

- Kidney failure serious medical problem
- Preferred treatment: kidney transplant
  - Cadaver kidneys
  - Donation from live patient
  - Must be blood- and tissue-type compatible
As of today (11/15/17):

116,319 people

waiting for a kidney transplant in the US.

http://optn.transplant.hrsa.gov

In 2008,

10,526 patients
received cadaver kidneys.

4,857 patients
received live donor kidneys.

5,920 patients died
or became too sick for a transplant.

(Hartline/Immortica)
Econ 101

• Trade for $$s

Legality

Section 301 of the National Organ Transplant Act, “Prohibition of organ purchases” imposes criminal penalties on any person who

“knowingly acquire[s], receive[s], or otherwise transfer[s] any human organ for valuable consideration for use in human transplantation”
Kidney exchanges (APD, UNOS, NKR)

1. Pairs register in database.
2. Form a graph, representing possible compatibilities
3. Feasible matchings found
4. Additional medical tests (“cross match”)
5. Transplants performed

Match Offer Process
KX via top-trading cycles?

• No: 0/1 rather than strict preferences
• Limits on cycle lengths

Kidney-Paired Donation

• Find vertex-disjoint cycles of length \( \leq k \) that cover as many vertices as possible

(Conitzer)
Special case: $k=2$

- If edges go in both directions, replace by undirected edge
- Remove other edges

![Diagram showing edges between nodes 1, 2, 3, 4, and 5.]

- Find *max cardinality matching* (max #edges, every vertex incident on at most one edge)
- Edmond’s algorithm (poly time.)

Complexity

- $k = 2$: in $P$ by Edmond’s alg
- $k = \infty$ in $P$
  (via a reduction to a maximum weight perfect matching problem)
K=\text{infinity}: \text{Edge formulation}

\begin{align*}
\max_y \quad & \sum_{(i,j) \in E} y_{ij} \\
\text{s.t.} \quad & \sum_j y_{ij} \leq 1, \quad \forall i \\
\quad & \sum_j y_{ij} = \sum_i y_{ji}, \quad \forall i
\end{align*}

Optimal solutions non-fractional!

\textbf{k > 2 (but not infinite)}

- For each edge from \(i\) to \(j\), make a binary variable \(x_{ij}\)
  - 1 if \(i\) gets \(j\)'s kidney, 0 otherwise
- maximize \(\sum_{ij} x_{ij}\)
- subject to:
  - for every \(i\): \(\Sigma_{j} x_{ij} = \Sigma_{j} x_{ji}\)
    - (number of kidneys received by \(i\) = number of kidneys given by \(i\))
  - for every \(j\): \(\Sigma_{i} x_{ij} \leq 1\)
    - (\(j\) gives at most 1 kidney)
  - for every path \(i_1 \ i_2 \ldots \ i_k \ i_{k+1}\) with \(i_1 \neq i_{k+1}\): \(\Sigma_{1 \leq j \leq k} x_{ij_{j+1}} \leq k-1\)
    - (no path of length \(k\) that doesn't end up where it started, hence no cycles greater than \(k\))
Complexity

• $k = 2$: in $P$ by Edmonds alg
• $k = \infty$ in $P$
• But $k = 3, 4, 5, \ldots$: $NP$-hard!

Cycle formulation (IP)

• $y_c \in \{0,1\}$ for each cycle
  \[
  \max \sum_c |c| \ y_c \\
  \text{s.t.} \sum_{c: i \in c} y_c \leq 1 \text{ for all pairs } i
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  \]

- For \( k=3, \ n=5000 \) pairs, still \( 4 \times 10^8 \) cycles

- Fix: Column generation ("branch and price")
  
  - Introduce new variables \( y_c \) on the fly
  - Solve up to 10,000 pairs, \( k=3 \)
Doing better

• Cycle size limited because of logistical and ethical concerns

• What can be done?

Altruistic Non-simultaneous Donor Chains
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• “Good Samaritan” donors. Enable (long) chains

• New computational challenge. Can reduce “cycles and chains” to “cycles” via zero weight back edges. Allow longer cycles, but ONLY if they involve an altruistic donor.

New clearing problem

• Can only solve 100 pairs with k<= 20
• Active research into this problem
  – Note: cannot use unrestricted length cycles, because may not include an altruistic donor
National Kidney Register

Completed 500+ transplants
Entirely “altruistic non-simultaneous chains”

• 60 person, 30 transplant chain (8/11- 12/11)

ref=health

Extensions

• Dynamics: deciding how to batch the clearing decision (more matches vs fairness
• Incentive considerations: How to get hospitals to share lists in order to promote better matching?
Other applications

- Peerflix (DVDs)
- Read It Swap It (books)
- Intervac (holiday houses)
- National odd shoe exchange

Announcements

- Monday 11/20: Guest lecture by Ravi Jagadeesan and Prof. Scott Kominers on two-sided matching
- Wednesday 11/29, in-class midterm 2
- Monday 11/27, 7-8:30pm, midterm 2 review, location TBD

- No more sections. We’ll announce office hours for the week of 11/27.