Lesson Plan

• Two-stage stochastic optimization
  – Stage one; stage two (recourse)
• Example: the Farmer’s problem, the Contractor’s problem
• Optimal stochastic solution
• EVPI and VSS
• Analytic solution method, Sample Average Approximation method

Reading: “A tutorial on stochastic programming,” Schapiro and Philpott, March 2007 (sections 1 and 2)
Stochastic Optimization

**MDP:**  \( M=(S,A,P,R) \)
- \( m \) states, \( n \) actions

Decision variables in the LP are \( \pi(s,a) \). Can only solve if \( m \cdot n \) is small.

But \( n \) may be large.

**Two-stage stochastic optimization** is special case for two periods.

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**Example: Contractor’s problem**

- Accept some projects at time 1. Gain revenue.
- \#actions = \# subsets of projects
- Exponentially large!

- At **time 2**, know the labor needed to complete each project. Can **recruit additional workers**.
Stochastic Optimization

**MDP:** $M=(S,A,P,R)$
- $m$ states, $n$ actions

Decision variables in the LP are $\pi(s,a)$. Can only solve if $m \times n$ is small.

But $n$ may be large (e.g., subsets of projects.)

$m$ may also be large (e.g., all things that can go wrong with a project!)

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**Two-stage stochastic optimization:**

Decision $x \in X$ at time step 1

**Random event** $z$: models *all* uncertainty (e.g., what could go wrong with every possible set of projects!)

Decision $y$ at time step 2.
Examples

• **T-shirt problem.** Buy $x$ t-shirts at time 1, cost $c$. At time 2, uncertain demand $z$ realized and need to either back-order or hold excess inventory.

• **Farmer’s problem.** Plant $x$ crops at time 1, incur cost. At time 2, uncertain weather $z$ realized. Buy/sell crops to ensure enough feed for animals, maximize profit.

• **Contractor’s problem.** Accept projects $x$ at time 1, get payments. At time 2, realize amount of labor $z$ needed for each project, recruit workers as necessary.

Two-stage stochastic optimization

![Diagram](attachment:image.png)

- **Stage 1:** Initial decision $X$
- **Stage 2:** Recourse decision $Y$

Events:
- $Event 1$: $y(x, z_1)$
- $Event 2$: $y(x, z_2)$
- $Event 3$: $y(x, z_3)$
Example: Farmer’s problem

(Birge & Louveaux’ 97)

- A farmer has 500 acres to plant wheat, grain and sugar beets. Needs min amount of wheat/corn to feed animals. Can also buy/sell in period 2.
- Without uncertainty about yield:

<table>
<thead>
<tr>
<th></th>
<th>Wheat</th>
<th>Corn</th>
<th>Beets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yield (T/acre)</td>
<td>2.5</td>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td>Cost ($/acre)</td>
<td>150</td>
<td>230</td>
<td>260</td>
</tr>
<tr>
<td>Sell price ($/T)</td>
<td>170</td>
<td>150</td>
<td>36 (under 6000 T) 10 (above 6000 T)</td>
</tr>
<tr>
<td>Purchase price ($/T)</td>
<td>238</td>
<td>210</td>
<td>-</td>
</tr>
<tr>
<td>Min requirement (T)</td>
<td>200</td>
<td>240</td>
<td>-</td>
</tr>
</tbody>
</table>

Formulation without Uncertainty

- Stage 1: \( x_1, x_2, x_3 = \) acres to plant for each crop
- Stage 2:
  - \( y_1^b, y_1^s = \) amount to buy, sell of crop 1 (w)
  - \( y_2^b, y_2^s = \) amount to buy, sell of crop 2 (c)
  - \( y_3^{s1}, y_3^{s2} = \) amount to sell of crop 3 at high, low price

\[
\begin{align*}
\text{max} & \quad -150x_1 - 230x_2 - 260x_3 \\
& \quad - 238y_1^b + 170y_1^s - 210y_2^b + 150y_2^s + 36y_3^{s1} + 10y_3^{s2} \\
\text{s.t.} & \quad x_1 + x_2 + x_3 \leq 500 \\
& \quad 2.5x_1 + y_1^b - y_1^s \geq 200 \\
& \quad 3x_2 + y_2^b - y_2^s \geq 240 \\
& \quad y_3^{sT} + y_3^{s2} \leq 20x_3 \\
& \quad y_3^{s1} \leq 6000 \\
& \quad x_1, x_2, \ldots, y_1^b, \ldots y_3^{s2} \geq 0
\end{align*}
\]
Optimal solution (No uncertainty)

<table>
<thead>
<tr>
<th></th>
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<th>Corn</th>
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</tr>
</thead>
<tbody>
<tr>
<td>acres</td>
<td>120</td>
<td>80</td>
<td>300</td>
</tr>
<tr>
<td>yield (T)</td>
<td>300</td>
<td>240</td>
<td>6000</td>
</tr>
<tr>
<td>sales</td>
<td>100</td>
<td>-</td>
<td>6000</td>
</tr>
<tr>
<td>purchase</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

High profit on sales

High profit up to 6000T

Feed animals

Introducing uncertainty

- **Weather** may be “good”, “normal” or “poor.”
  This affects the yield on each crop
- Given this **uncertainty**, what is optimal **decision in stage one** about which crops to plant?
Warm-up: An omniscient farmer

Suppose the farmer can predict the weather…

<table>
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<th></th>
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<th>Corn</th>
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</tr>
</thead>
<tbody>
<tr>
<td><strong>acres</strong></td>
<td>183.3</td>
<td>66.7</td>
<td>250</td>
</tr>
<tr>
<td>yield (T)</td>
<td>550</td>
<td>240</td>
<td>6000</td>
</tr>
<tr>
<td>sales</td>
<td>350</td>
<td>-</td>
<td>6000</td>
</tr>
<tr>
<td>purchase</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Optimal decision if weather good

<table>
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<th>Beets</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>acres</strong></td>
<td>120</td>
<td>80</td>
<td>300</td>
</tr>
<tr>
<td>yield (T)</td>
<td>300</td>
<td>240</td>
<td>6000</td>
</tr>
<tr>
<td>sales</td>
<td>100</td>
<td>-</td>
<td>6000</td>
</tr>
<tr>
<td>purchase</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Optimal decision if weather normal

<table>
<thead>
<tr>
<th></th>
<th>Wheat</th>
<th>Corn</th>
<th>Beets</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>acres</strong></td>
<td>100</td>
<td>25</td>
<td>375</td>
</tr>
<tr>
<td>yield (T)</td>
<td>200</td>
<td>60</td>
<td>6000</td>
</tr>
<tr>
<td>sales</td>
<td>-</td>
<td>-</td>
<td>6000</td>
</tr>
<tr>
<td>purchase</td>
<td>-</td>
<td>180</td>
<td>-</td>
</tr>
</tbody>
</table>
Comparing the Omniscient solutions

<table>
<thead>
<tr>
<th></th>
<th>good</th>
<th>normal</th>
<th>poor</th>
</tr>
</thead>
<tbody>
<tr>
<td>wheat (acres)</td>
<td>183.3</td>
<td>120</td>
<td>100</td>
</tr>
<tr>
<td>corn (acres)</td>
<td>66.7</td>
<td>80</td>
<td>25</td>
</tr>
<tr>
<td>beets (acres)</td>
<td>250</td>
<td>300</td>
<td>375</td>
</tr>
</tbody>
</table>

A lot of variation: best to plant between 183 and 100 acres of wheat, depending on the weather.

What to do?!

Two-stage Stochastic Optimization

- Uncertain event. Weather is {good, normal, poor}
- Probability 1/3 on each
- What to do in period one, and in period two (recourse)?
\[ \text{max} \ -150x_1 - 230x_2 - 260x_3 \\
+1/3 \ (-238y_{11}^b + 170y_{11}^s - 210y_{21}^b + 150y_{21}^s + 36y_{31}^{s1} + 10y_{31}^{s2}) \\
+1/3 \ (-238y_{12}^b + 170y_{12}^s - 210y_{22}^b + 150y_{22}^s + 36y_{32}^{s1} + 10y_{32}^{s2}) \\
+1/3 \ (-238y_{13}^b + 170y_{13}^s - 210y_{23}^b + 150y_{23}^s + 36y_{33}^{s1} + 10y_{33}^{s2}) \]

\[ \begin{align*}
x_1 + x_2 + x_3 & \leq 500 \\
3x_1 + y_{11}^b - y_{11}^s & \geq 200 \\
3.6x_2 + y_{21}^b - y_{21}^s & \geq 240 \\
y_{31}^{s1} + y_{31}^{s2} & \leq 24 \\
y_{31}^s & \leq 6000 \\
2.5x_1 + y_{12}^b - y_{12}^s & \geq 200 \\
3x_2 + y_{22}^b - y_{22}^s & \geq 240 \\
y_{32}^{s1} + y_{32}^{s2} & \leq 20 \\
y_{32}^s & \leq 6000 \\
2x_1 + y_{13}^b - y_{13}^s & \geq 200 \\
2.4x_2 + y_{23}^b - y_{23}^s & \geq 240 \\
y_{33}^{s1} + y_{33}^{s2} & \leq 16 \\
y_{33}^s & \leq 6000 \\
x_1, x_2, \ldots, y_{11}^b, \ldots, y_{33}^{s2} & \geq 0
\end{align*} \]
Optimal solution with Uncertainty

<table>
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<tr>
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<th><strong>Corn</strong></th>
<th><strong>Beets</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Acres</strong></td>
<td>170</td>
<td>80</td>
<td>250</td>
</tr>
<tr>
<td><strong>Yield (T)</strong></td>
<td>510</td>
<td>288</td>
<td>6000</td>
</tr>
<tr>
<td><strong>Sales (T)</strong></td>
<td>310</td>
<td>48</td>
<td>6000</td>
</tr>
<tr>
<td><strong>Purchase (T)</strong></td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td><strong>Yield (T)</strong></td>
<td>425</td>
<td>240</td>
<td>5000</td>
</tr>
<tr>
<td><strong>Sales (T)</strong></td>
<td>225</td>
<td>-</td>
<td>5000</td>
</tr>
<tr>
<td><strong>Purchase (T)</strong></td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td><strong>Yield (T)</strong></td>
<td>340</td>
<td>192</td>
<td>4000</td>
</tr>
<tr>
<td><strong>Sales (T)</strong></td>
<td>140</td>
<td>-</td>
<td>4000</td>
</tr>
<tr>
<td><strong>Purchase (T)</strong></td>
<td>-</td>
<td>48</td>
<td>-</td>
</tr>
</tbody>
</table>

Profit: $108,390

“omniscient solution” (if knew weather) would bring expected value $\frac{1}{3}(167+118+59) \approx $114,667

Expected value of Perfect Information (EVPI): $114,667 - 108,390 = $6,277

Value of Stochastic Solution (VSS): $108,390 - 107,240 = $1,150
EVPI and VSS

- $C(x)$: stage 1 value of stage 1 decision
- $Q(x,z)$: stage 2 value of opt recourse given $(x,z)$
- $V(z)$: total value of optimal stage 1 and 2 decisions given $z$

- EVPI: expected value of perfect information
  $$E_z[V(z)] – \max_x (C(x) + E_z[Q(x,z)])$$(1)

- VSS: expected value of stochastic solution
  $$\max_x (C(x) + E_z[Q(x,z)]) – (C(x^*) + E_z[Q(x^*,z)])$$
  where $x^*$ is opt “in expectation” stage 1 decision

Two-Stage Stochastic Programming

- First-stage decision $x \in \mathbb{R}^n$. Event $z=(q,T,W,h)$ defines second-stage problem.
- $y \in \mathbb{R}^m$ second-stage decision

First-stage problem:
$$\max_c \ c^\top x + E_z[Q(x,z)]$$
$$\text{s.t. } Ax \leq b$$
$$\quad x \geq 0$$

Second-stage problem:
$$Q(x,z) = \max_y q^\top y$$
$$\text{s.t. } Tx + Wy \leq h$$
$$\quad y \geq 0$$

where $E_z[Q(x,z)]$ is the expected value of $Q(x,z)$.
Example: Farmer’s problem

- Only $T$ matrix is uncertain in the farmer’s problem
- Let $t_i$ denote yield of crop $i$; realized event $z=(t_1, t_2, t_3)$
- Second-stage decision problem:

$$Q(x, z) = \max -238 y_1^b + 170 y_1^s - 210 y_2^b + 150 y_2^s + 36 y_3^{s_1} + 10 y_3^{s_2}$$

s.t.

\begin{align*}
t_1 x_1 + y_1^b - y_1^s & \geq 200 \\
t_2 x_2 + y_2^b - y_2^s & \geq 240 \\
y_3^{s_1} + y_3^{s_2} & \leq t_3 x_3 \\
y_3^{s_1} & \leq 6000
\end{align*}

Maximize period 2 value given $(x, z)$.

Computational approaches

1. **Small number of possible events**, can form a single LP and solve (e.g., Farmer’s problem.)

2. Event $z$ is a continuous r.v., but can solve for $E_z[Q(x,z)]$ analytically and then solve for stage 1 analytically.

3. Cannot use approach (1) or (2). Use **sample average approximation (SAA).**
Analytical approach: Example

• We’re giving away Harvard-Yale T-shirts!
• Must decide how many to order \( x \) at cost \( c = $1 \).
  Demand \( z \sim U(0,100) \) uncertain
• Cost \( b = $1.5 \) for “backorder cost” if \( z > x \), holding
  cost \( h = $0.1 \) if \( x > z \).

\[
\min cx + E_z[Q(x,z)]
\]
\[
Q(x,z) = b \max(z-x,0) + h \max(x-z,0)
\]

Analytical approach: Example

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\[
\min cx + E_z[Q(x,z)]
\]
\[
Q(x,z) = b \max(z-x,0) + h \max(x-z,0)
\]

• Write \( E_z[Q(x,z)] = W(x) \)
• Solve for \( x^* \) by first-order optimality: \( c + W'(x) = 0 \)
Example: \( cx + Q(x,z) \) for \( z=50 \)

\[
\begin{align*}
\text{cx } + Q(x,50) &= 1x + 1.5 \max(50-x,0) + 0.1 \max(x-50,0)
\end{align*}
\]

- \( \text{cx } + E_z[Q(x,z)] = bE_z[z] + (c-b)x + (b+h) \int_0^x F(z) \, dz \)
  
  \[
  = (1.5)(50) + (-0.5)x + 1.6 \int_0^x z/100 \, dz
  \]
  
  \[
  = 75 - 0.5x + 0.008x^2
  \]
\[ cx + E_z[Q(x, z)] \]

**Computational approaches**

1. **Small number of possible events**, can form a single LP and solve (e.g., Farmer’s problem.)

2. Event \( z \) is a continuous r.v., but can solve for \( E_z[Q(x, z)] \) analytically and then solve for stage 1 analytically.

3. Cannot use approach (1) or (2). Use **sample average approximation** (SAA).
Sample Average Approximation

(Kleywegt et al. 2001)

- **Approximate** $E_z[Q(x,z)]$ by sampling $K$ possible events, with
  
  $$E_z[Q(x,z)] \approx \frac{1}{K} \sum_{k=1}^{K} Q(x, z_k)$$

- Solve:
  $$\min_{x, y} \quad c^T x + \frac{1}{K} \sum_k q_k^T y_k$$
  s.t. $T_k x + W_k y_k \leq h_k$ for all $k$

Example: T-shirt problem

- Consider two realized (demand) events:
  $$z_1 = 20, z_2 = 80$$

- Objective: $c x + E_z[Q(x,z)] = c x + \frac{1}{2}Q(x,z_1) + \frac{1}{2}Q(x,z_2)$

- Have one set of period 2 decision variables per sampled event

- Can also consider three (or more!) scenarios…
Example: Approximating $c x + E_z[Q(x,z)]$

Theoretical properties of SAA
(Kleywegt et al. 2001)

- $V_K = \text{exp. value of approx solution for } K \text{ samples}$
- $S_K^d = \text{all solutions within distance } d \text{ of approx soln}$
- $V^* = \text{exp value of optimal solution}$
- $S^d = \text{all solutions within distance } d \text{ of optimal soln}$

**Theorem:**

1. $\lim_{K \to \infty} V_K = V^*$
2. $S_K^d \subseteq S^d$ with probability 1 as $K \to \infty$, for any distance $d$. 
Example Applications

(Kleywegt et al. 01; van Hentenryck)

- Contractor’s problem.
- Airline crew scheduling. Assign crew to routes, but each route takes an uncertain time. May need to pay over-time.
- Power restoration (schedule repair crews, schedule power restoration. Minimize overall size of blackout.)
- Scheduling evacuations (save as many lives as possible.)

Contractor’s Problem

(Kleywegt et al. 01)

20 projects. Fraction of solutions within delta (d) of optimal:
Summary: Stochastic Optimization

• Scales up MDPs to settings with many decisions, many possible states

• Solve first-stage in anticipation of second-stage (recourse).
• Find value-maximizing first-stage decision
• “Optimal stochastic solution”

• Sample average approximation method

Announcements

• Midterm 2, Monday Dec. 2, in-class
  – Non-cumulative
  – Covers Dual Simplex, IP, Markov chains and MDP
  – Doesn’t cover today’s material