AM 121: Intro to Optimization Models and Methods

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Lecture 9: Complementary Slackness

Yiling Chen
SEAS

Lesson Plan

- **Complementary slackness**: a compatibility requirement between an optimal primal and an optimal dual solution.
- Stated for the standard inequality form.
- Used to reason about optimality, build intuitions. Motivates new family of algorithms.

Jensen & Bard: P127-128

Complementary Slackness

(standard inequality form)

$$\max\ c^T x \qquad \text{s.t } Ax \le b \qquad x \ge 0$$

$$\min\ b^T y \qquad \text{s.t } A^T y \ge c \qquad y \ge 0$$

- Vars dual <-> constr. primal. Constr. dual <-> vars in primal
- vars in one problem are "complementary" to constr. in other

- An inequality constraint "has slack" if it is not binding. A non-negativity constraint "has slack" if the variable is positive.
- **Complementary slackness:** cannot be slack in both a constraint and its associated variable.

- For example, if primal var > 0 then the dual constr. Is binding. If primal constr. not binding, then dual var = 0.

Complementary Slackness

(standard inequality form)

$$\max\ c^T x \qquad \text{s.t } Ax \le b \qquad x \ge 0$$

$$\min\ b^T y \qquad \text{s.t } A^T y \ge c \qquad y \ge 0$$

- Let $s_1, \ldots s_m$ denote primal slack and $e_1, \ldots e_n$ denote dual excess.
- **Theorem**. A primal feasible *x* and a dual feasible *y* are both **optimal** if and only if

$$x_j e_j = 0 \quad j = 1, \ldots, n$$

$$y_i s_i = 0 \quad i = 1, \ldots, m$$

- (primal var, dual constr.), (dual var, primal constr.)
- **"Cannot be slack in both."**

Intuition

$$\max\ c^T x \quad \text{s.t } Ax \le b,\ x \ge 0$$

$$\min\ b^T y \quad \text{s.t } A^T y \ge c,\ y \ge 0$$

- Primal slack $s_1, \ldots s_m$
- Complementary slackness:

$$y_i s_i = 0 \quad i = 1, \ldots, m$$

(dual var, primal constraint)

- If dual var positive, then primal constraint binding. (*Shadow price > 0 => scarce resources*).
- If primal constraint not binding, then dual var = 0. (*Surplus resources => shadow price = 0*.)

Proof. By definition of dual and primal, we have:

$$c^T x \le (A^T y)^T x = y^T Ax \le y^T b = b^T y$$

- First inequality can be written as:

$$\sum_j c_j x_j \le \sum_j (\sum_i a_{ij} y_i) x_j \qquad (*)$$

- In particular

$$c_j \le \sum_i a_{ij} y_i$$

- If x solves (**P**) and y solves (**D**), $c^T x = b^T y$ {strong duality} and (*) must be an equality.
- Since $x_j >= 0$ (all j), for equality we need:

$$x_j > 0 \Longrightarrow c_j = \sum_i a_{ij} y_i \qquad j = 1, \ldots, n$$

$$\text{and } (c_j < \sum_i a_{ij} y_i) \Longrightarrow x_j = 0 \qquad j = 1, \ldots, n$$

- Equivalently:

$$x_j (\sum_i a_{ij} y_i - c_j) = 0 \qquad \forall j = 1, \ldots, n$$

$$\text{or, } \ x_j e_j = 0 \qquad \forall j = 1, \ldots, n$$

- This is the first CS condition.

Proof. By definition of dual and primal, we have:

$$c^T x \leq (A^T y)^T x = y^T Ax \leq y^T b = b^T y$$

- Similar analysis establishes $y^T Ax = y^T b$ if and only if $y_i s_i = 0 \;\; \forall i = 1, \ldots, m$

Complementary Slackness

- $x_j e_j = 0 \;\; j = 1, \ldots, n$

  (jth primal var>0) $\Longrightarrow$ (jth dual constr. binds) (1)

  (jth dual constr. slack) $\Longrightarrow$ (jth primal var = 0) (2)

  Note: $(1) \equiv (2)$

- $y_i s_i = 0 \;\; i = 1, \ldots, m$

  (ith dual var > 0) $\Longrightarrow$ (ith primal constr. binds (3)

  (jth primal constr. slack) $\Longrightarrow$ (ith dual var = 0) (4)

  Note: $(3) \equiv (4)$

- Can work with whichever of (1) or (2), and (3) or (4) are most convenient

Example: Comp. Slackness

- max $x_1 - x_2$ s.t. $-3x_1 + x_2 \leq 2$, $2x_1 - x_2 \leq -1$, $x_1, x_2 \geq 0$

  min $2y_1 - y_2$ s.t. $-3y_1 + 2y_2 \geq 1$, $y_1 - y_2 \geq -1$, $y_1, y_2 \geq 0$

(1) If $x^*_1 > 0$ in optimal primal solution, then first constraint in the optimal dual solution y^* is binding.

(4) If first constraint in optimal primal solution x^* is not binding, then variable $y_1 = 0$ in optimal dual solution.

Careful: OK for both (var, constraint) pair to have zero slack;
i.e. variables zero and constraint binding
NOT OK for variable to be positive and constraint slack.

Restricted Dual

- **Definition**. The **restricted dual RD(x) problem** given a feasible primal solution x modifies dual to impose CS conditions with respect to x in addition to dual feasibility.
- Any objective function can be adopted.

- **Theorem.** A feasible primal x is optimal if and only if the restricted dual **RD(x)** has a feasible solution.
- **Proof.**
- **(=>)** x optimal => the optimal y satisfies CS with x => RD(x) is feasible.
- **(<=) RD(x)** is feasible => exists a dual solution y satisfying CS with x => both y and x are optimal.

Example of RD (Furniture)

- Recall optimal $x^*=(2, 0, 8, 24, 0, 0)$, $z=280$
- Use (1) and (4) to impose constraints on dual, find feasible dual soln that satisfies CS:

$$x^*_1 > 0 \Longrightarrow e_1 = 0$$
$$x^*_3 > 0 \Longrightarrow e_3 = 0$$
$$s^*_1 > 0 \Longrightarrow y_1 = 0$$

- The dual LP is:

$$\begin{aligned} \min\ & 48y_1 + 20y_2 + 8y_3 \\ \text{s.t. } & 8y_1 + 4y_2 + 2y_3 \geq 60 \\ & 6y_1 + 2y_2 + 1.5y_3 \geq 30 \\ & y_1 + 1.5y_2 + 0.5y_3 \geq 20 \\ & y_1, \quad y_2, \quad y_3 \geq 0 \end{aligned}$$

- The **RD(x*)** is:

(since $e_1=0$, $y_1=0$) $4y_2+2y_3=60$

(since $e_3=0$, $y_1=0$) $1.5y_2+0.5y_3=20$

(since $y_1=0$) $2y_2+1.5y_3 \geq 30$

Feasible $y=(0, 10, 10, 0, 15, 0)$
So, y and x^* are optimal.

Primal-Dual Algorithms

- Can interpret the simplex method as maintaining feasible primal and an infeasible dual that satisfies CS conditions with primal. Terminates when dual is feasible => primal and dual are optimal.
- See this next lecture (along with the **dual simplex algorithm**).

Properties maintained in each iteration

	P feasible	D feasible	CS
P simplex	√	x	√
D simplex	x	√	√
primal-dual	√	√	x

Summary: Complementary Slackness

- Stated for the standard inequality form.
- (P var, D con) (D var, P con). Cannot both have slack. In particular:
  - If primal var > 0, then associated dual inequality must be binding.
  - If dual var > 0, then associated primal inequality must be binding.
- Can also state the contrapositives:
  - If dual inequality not binding, then primal var zero
  - If primal inequality not binding, then dual var zero
- The restricted dual **(RD)**:
  - Given primal *x*, impose CS, look for a feas dual
  - **Yes**: *x* is optimal. **No**: *x* is not optimal.