Lecture 9: Complementary Slackness

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Lesson Plan

• **Complementary slackness**: a compatibility requirement between an optimal primal and an optimal dual solution.

• Stated for the standard inequality form.

• Used to reason about optimality, build intuitions. Motivates new family of algorithms.

Jensen & Bard: P127-128
Complementary Slackness

(standard inequality form)

\[
\begin{align*}
\text{max} & \quad c^T x & \text{min} & \quad b^T y \\
\text{s.t} & \quad Ax \leq b & \text{s.t} & \quad A^T y \geq c \\
& \quad x \geq 0 & & \quad y \geq 0
\end{align*}
\]

- Vars dual <-> constr. primal. Constr. dual <-> vars in primal
- vars in one problem are “complementary” to constr. in other

- An inequality constraint “has slack” if it is not binding. A non-negativity constraint “has slack” if the variable is positive.
- **Complementary slackness:** cannot be slack in both a constraint and its associated variable.

- For example, if primal var > 0 then the dual constr. Is binding. If primal constr. not binding, then dual var = 0.

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Complementary Slackness

(standard inequality form)

\[
\begin{align*}
\text{max} & \quad c^T x & \text{min} & \quad b^T y \\
\text{s.t} & \quad Ax \leq b & \text{s.t} & \quad A^T y \geq c \\
& \quad x \geq 0 & & \quad y \geq 0
\end{align*}
\]

- Let \( s_1, \ldots s_m \) denote primal slack and \( e_1, \ldots e_n \) denote dual excess.
- **Theorem.** A primal feasible \( x \) and a dual feasible \( y \) are both **optimal** if and only if

  \[
  x_j e_j = 0 \quad j = 1, \ldots, n \\
  y_i s_i = 0 \quad i = 1, \ldots, m
  \]

- (primal var, dual constr.), (dual var, primal constr.)
- “Cannot be slack in both.”
Intuition

\[
\begin{align*}
\max & \quad c^T x \\
\text{s.t} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\min & \quad b^T y \\
\text{s.t} & \quad A^T y \geq c \\
& \quad y \geq 0
\end{align*}
\]

- Primal slack \( s_1, \ldots, s_m \)
- Complementary slackness:
  \( y_i s_i = 0 \quad i = 1, \ldots, m \)  
  (dual var, primal constraint)

- If dual var positive, then primal constraint binding.  
  (\textit{Shadow price > 0} \implies \textit{scarce resources}).
- If primal constraint not binding, then dual var = 0.  
  (\textit{Surplus resources} \implies \textit{shadow price = 0}.)

\textbf{Proof.} By definition of dual and primal, we have:

\[
c^T x \leq (A^T y)^T x = y^T A x \leq b^T y
\]

- First inequality can be written as:
  \[
  \sum_j c_j x_j \leq \sum_j (\sum_i a_{ij} y_i) x_j  
  \]
  (*)
- In particular
  \[
  c_j \leq \sum_i a_{ij} y_i
  \]
- If \( x \) solves (P) and \( y \) solves (D), \( c^T x = b^T y \) \{strong duality\} and (*) must be an equality.
- Since \( x_j \geq 0 \) (all \( j \)), for equality we need:
  \[
  x_j > 0 \implies c_j = \sum_i a_{ij} y_i \quad j = 1, \ldots, n
  \]
  and \( (c_j < \sum_i a_{ij} y_i) \implies x_j = 0 \quad j = 1, \ldots, n \)
- Equivalently:
  \[
  x_j (\sum_i a_{ij} y_i - c_j) = 0 \quad \forall j = 1, \ldots, n
  \]
  or\(, \quad x_j c_j = 0 \quad \forall j = 1, \ldots, n \)
- This is the first CS condition.
Proof. By definition of dual and primal, we have:
\[ c^T x \leq (A^T y)^T x = y^T A x \leq y^T b = b^T y \]

- Similar analysis establishes \( y^T A x = y^T b \) if and only if \( y_i s_i = 0 \ \forall i = 1, \ldots, m \)

Complementary Slackness

- \( x_j e_j = 0 \ j = 1, \ldots, n \)  
  (jth primal var > 0) \( \implies \) (jth dual constr. binds) \ (1)  
  (jth dual constr. slack) \( \implies \) (jth primal var = 0) \ (2)

Note: \( (1) \equiv (2) \)

- \( y_i s_i = 0 \ i = 1, \ldots, m \)  
  (ith dual var > 0) \( \implies \) (ith primal constr. binds) \ (3)  
  (jth primal constr. slack) \( \implies \) (ith dual var = 0) \ (4)

Note: \( (3) \equiv (4) \)

- Can work with whichever of (1) or (2), and (3) or (4) are most convenient
Example: Comp. Slackness

- max $x_1 - x_2$  \quad min $2y_1 - y_2$
- s.t. $-3x_1 + x_2 \leq 2$  \quad s.t. $-3y_1 + 2y_2 \geq 1$
  $2x_1 - x_2 \leq -1$  \quad $y_1 - y_2 \geq -1$
  $x_1, x_2 \geq 0$  \quad $y_1, y_2 \geq 0$

(1) If $x^*_1 > 0$ in optimal primal solution, then first constraint in the optimal dual solution $y^*$ is binding.

(4) If first constraint in optimal primal solution $x^*$ is not binding, then variable $y_1 = 0$ in optimal dual solution.

**Careful**: OK for both (var, constraint) pair to have zero slack; i.e. variables zero and constraint binding NOT OK for variable to be positive and constraint slack.

Restricted Dual

- **Definition.** The **restricted dual RD(x) problem** given a feasible primal solution $x$ modifies dual to impose CS conditions with respect to $x$ in addition to dual feasibility.

- Any objective function can be adopted.

- **Theorem.** A feasible primal $x$ is optimal if and only if the restricted dual $\text{RD}(x)$ has a feasible solution.

- **Proof.**
  - $(\Rightarrow)$ $x$ optimal => the optimal $y$ satisfies CS with $x$ => $\text{RD}(x)$ is feasible.
  - $(\Leftarrow)$ $\text{RD}(x)$ is feasible => exists a dual solution $y$ satisfying CS with $x$ => both $y$ and $x$ are optimal.
Example of RD (Furniture)

- Recall optimal \( x^* = (2, 0, 8, 24, 0, 0), z = 280 \)
- Use (1) and (4) to impose constraints on dual, find feasible dual soln that satisfies CS:
  - \( x^*_1 > 0 \implies e_1 = 0 \)
  - \( x^*_3 > 0 \implies e_3 = 0 \)
  - \( s^*_1 > 0 \implies y_1 = 0 \)
- The dual LP is:
  \[
  \begin{align*}
  \text{min} & \quad 48y_1 + 20y_2 + 8y_3 \\
  \text{s.t.} & \quad 8y_1 + 4y_2 + 2y_3 \geq 60 \\
  & \quad 6y_1 + 2y_2 + 1.5y_3 \geq 30 \\
  & \quad y_1 + 1.5y_2 + 0.5y_3 \geq 20 \\
  & \quad y_1, y_2, y_3 \geq 0
  \end{align*}
  \]
- The RD\((x^*)\) is:
  \[
  \begin{align*}
  & \text{(since } e_1 = 0) \quad 4y_2 + 2y_3 = 60 \\
  & \text{(since } e_2 = 0) \quad 1.5y_2 + 0.5y_3 = 20 \\
  & \text{(since } y_1 = 0) \quad 2y_2 + 1.5y_3 \geq 30
  \end{align*}
  \]
  Feasible \( y = (0, 10, 10, 0, 15, 0) \)
  So, \( y \) and \( x^* \) are optimal.

Primal-Dual Algorithms

- Can interpret the simplex method as maintaining feasible primal and an infeasible dual that satisfies CS conditions with primal. Terminates when dual is feasible => primal and dual are optimal.
- See this next lecture (along with the dual simplex algorithm).

\[
\begin{array}{|c|c|c|c|}
\hline
 & \text{P feasible} & \text{D feasible} & \text{CS} \\
\hline
\text{P simplex} & \checkmark & \times & \checkmark \\
\hline
\text{D simplex} & \times & \checkmark & \checkmark \\
\hline
\text{primal-dual} & \checkmark & \checkmark & \times \\
\hline
\end{array}
\]

Properties maintained in each iteration
Summary: Complementary Slackness

• Stated for the standard inequality form.
• \((P \text{ var}, D \text{ con}) (D \text{ var}, P \text{ con})\). Cannot both have slack. In particular:
  – If \(\text{primal var} > 0\), then associated dual inequality must be binding.
  – If \(\text{dual var} > 0\), then associated primal inequality must be binding.
• Can also state the contrapositives:
  – If dual inequality not binding, then primal var zero
  – If primal inequality not binding, then dual var zero
• The restricted dual (RD):
  – Given primal \(x\), impose CS, look for a feas dual
  – **Yes**: \(x\) is optimal. **No**: \(x\) is not optimal.