Lecture 9: Complementary Slackness

Lesson Plan

• Complementary slackness
• A compatibility requirement between an optimal primal and an optimal dual solution
• Can be used to reason about optimality, build intuitions
• Also motivates a new family of algorithms

Jensen & Bard: P127-128
Complementary Slackness

\[
\begin{align*}
\text{max} & \quad c^T x & \text{min} & \quad b^T y \\
\text{s.t} & \quad Ax \leq b & \text{s.t} & \quad A^T y \geq c \\
& \quad x \geq 0 & & \quad y \geq 0
\end{align*}
\]

• Consider primal slack var’s \( s_1, \ldots s_m \) and dual surplus var’s \( e_1, \ldots e_n \)

• **Theorem.** A primal feasible \( x \) and a dual feasible \( y \) are both **optimal** if and only if

\[
\begin{align*}
x_j e_j &= 0 & j &= 1, \ldots, n \\
y_i s_i &= 0 & i &= 1, \ldots, m
\end{align*}
\]

• If \( x_j > 0 \) then corresponding inequality in dual is binding
• If \( y_i > 0 \) then corresponding inequality in primal is binding

Economic Intuition

\[
\begin{align*}
\text{max} & \quad c^T x & \text{min} & \quad b^T y \\
\text{s.t} & \quad Ax \leq b & \text{s.t} & \quad A^T y \geq c \\
& \quad x \geq 0 & & \quad y \geq 0
\end{align*}
\]

• Primal slack var’s \( s_1, \ldots s_m \) and dual surplus var’s \( e_1, \ldots e_n \)

\[
y_i s_i = 0 \quad i = 1, \ldots, m
\]

• If there is slack in the primal resource constraint, then dual ("shadow price") should be zero
• If the shadow price is non-zero, then there must be scarce resources (no slack in primal).
• **Proof.** \( c^T x \leq (A^T y)^T x = y^T Ax \leq y^T b = b^T y \)
• The first inequality can be written
\[
\sum_j c_j x_j \leq \sum_i \left( \sum_i a_{ij}y_i \right) x_j
\]
• These terms are **equal** if and only if
\[
x_j > 0 \implies c_j = \sum_i a_{ij}y_i \quad j = 1, \ldots, n
\]
and \( c_j < \sum_i a_{ij}y_i \implies x_j = 0 \quad j = 1, \ldots, n \)
• Equivalent to requiring
\[
x_j \left( \sum_i a_{ij}y_i - c_j \right) = 0 \quad \forall j = 1, \ldots, n
\]
or, \( x_j e_j = 0 \quad \forall j = 1, \ldots, n \)
• Similar analysis establishes \( y^T A x = y^T b \) if and only if \( y_i s_i = 0 \quad \forall i = 1, \ldots, m \)

### Complementary Slackness

- \( x_j e_j = 0 \quad j = 1, \ldots, n \)
  - (jth primal var>0) \implies (jth dual surplus=0) \quad (1)
  - (jth dual surplus>0) \implies (jth primal var=0) \quad (2)

**Note:** (1) \( \equiv \) (2)
- \( y_i s_i = 0 \quad i = 1, \ldots, m \)
  - (ith dual var > 0) \implies (ith primal slack = 0) \quad (3)
  - (ith primal slack > 0) \implies (ith dual var = 0) \quad (4)

**Note:** (3) \( \equiv \) (4)

- We can use whichever of (1) or (2), and (3) or (4) are most convenient
Simple Example

- \[\max x_1 - x_2\] \[\min 2y_1 - y_2\]
- \[s_1 \text{ s.t. } -3x_1 + x_2 \leq 2 \] \[e_1 \text{ s.t. } -3y_1 + 2y_2 \geq 1\]
- \[s_2 \quad 2x_1 - x_2 \leq -1 \] \[e_2 \quad y_1 - y_2 \geq -1\]
- \[x_1, x_2 \geq 0 \] \[y_1, y_2 \geq 0\]

(1) If \(x_1 > 0\) in optimal primal, then \(e_1 = 0\), and first constraint in optimal dual (corresponds to variable 1) is binding.

(4) If \(s_1 > 0\) in optimal primal, then \(y_1 = 0\), and first variable in optimal dual (corresponds to constraint 1) is zero.

Careful: if \(s_1 = 0\), then \(y_1\) can be 0 or positive. Only make inference if inequality is non-binding, or a variable is non-zero.

Furniture Example

- Furniture example: \(x^*=(2, 0, 8, 24, 0, 0)\), \(z=280\)
- Use (1) and (4) to impose constraints on dual, look for a feasible dual sol’n that satisfies CS
  \[x_1 > 0 \implies e_1 = 0\]
  \[x_3 > 0 \implies e_3 = 0\]
  \[s_1 > 0 \implies y_1 = 0\]
- The dual problem is:
  \[\min 48y_1 + 20y_2 + 8y_3\]
  \[\text{s.t. } 8y_1 + 4y_2 + 2y_3 \geq 60\]
  \[6y_1 + 2y_2 + 1.5y_3 \geq 30\]
  \[y_1 + 1.5y_2 + 0.5y_3 \geq 20\]
  \[y_1, y_2, y_3 \geq 0\]

- With CS constraints:
  (since \(e_1=0, y_1=0\)) \(4y_2 + 2y_3 = 60\)
  (since \(e_3=0, y_1=0\)) \(1.5y_2 + 0.5y_3 = 20\)
  (since \(y_1=0\)) \(2y_2 + 1.5y_3 \geq 30\)

Feasible \(y^*=(0, 10, 10, 0, 15, 0)\) satisfies CS with \(x^*\); so both \(y^*\) and \(x^*\) are optimal.

(restricted dual problem)
Other uses of Comp Slackness

- Disproving optimality of a solution.
  - Given a feasible primal solution \( x \), disprove optimality by showing there is no feasible solution to the “restricted dual” problem
  - Restricted dual takes \( x \), modifies the inequalities in the dual to impose CS conditions with respect to \( x \), and looks for a feasible dual satisfying the CS conditions
  - Can adopt any objective in solving the restricted dual

Other uses of Comp Slackness

- **Primal-dual algorithms**
- Can interpret the (primal) simplex algorithm as maintaining a dual solution that satisfies CS conditions with the primal feasible solution, and terminating when the dual solution is feasible.
- Next lecture (along with dual simplex algorithm).

<table>
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<tr>
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<th>P feasible</th>
<th>D feasible</th>
<th>CS</th>
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<tr>
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<td>x</td>
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<tr>
<td>D simplex</td>
<td>x</td>
<td>✓</td>
<td>✓</td>
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<tr>
<td>primal-dual</td>
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<td>✓</td>
<td>x</td>
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Summary: Comp. Slackness

• Standard inequality form
  – If primal var > 0, that dual inequality binding
  – If dual var > 0, then primal inequality binding

• Contrapositives:
  – If dual inequality loose, then primal var zero
  – If primal inequality loose, then dual var zero

• Restricted dual:
  – Given primal $x$, impose CS, look for a feas dual
  – Yes: $x$ is optimal. No: $x$ is not optimal.

• New algorithms