AM 121: Intro to Optimization Models and Methods

Fall 2019

Lecture 8: Sensitivity Analysis

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SEAS

Lesson Plan: Sensitivity

- Explore effect of changes in obj coefficients, and constraints on the optimal solution
  - see a connection to duality
- Approaches:
  - Geometric intuition
  - Use AMPL to get sensitivity information
  - Use “basis to tableau” eqns

Jensen & Bard: 4.1

Sensitivity Analysis

- What happens if the data change slightly?
  - E.g., a change in right-hand side value
  - E.g., a change in objective coefficient
  - E.g., a new decision variable
  - E.g., modifying the entries in the column corresponding to a variable

- Important to understand the robustness of a solution.

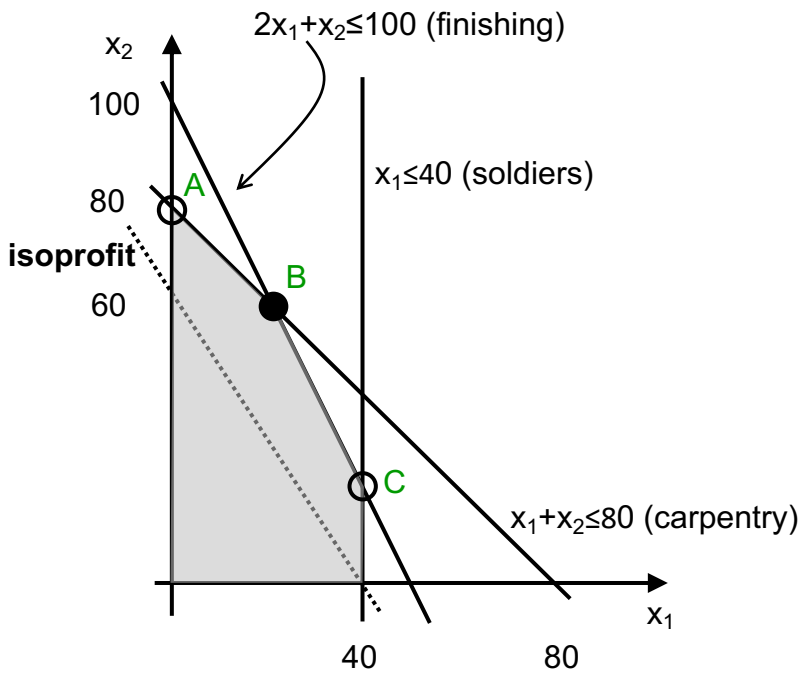
First Approach: Graphical Intuition

Example 1: Wooden toys

- Make toy soldiers and toy trains
- $3 profit per soldier, $2 profit per train
- Skilled labor of two types (carpentry and finishing)
  - Soldier: 2 hours finishing, 1 hour carpentry
  - Train: 1 hour finishing, 1 hour carpentry
  - 100 finishing hours, 80 carpentry hours each week
  - Unlimited demand for trains, at most 40 soldiers purchased each week
- Goal: maximize profit
- (Use LP, assume solution will be integral)

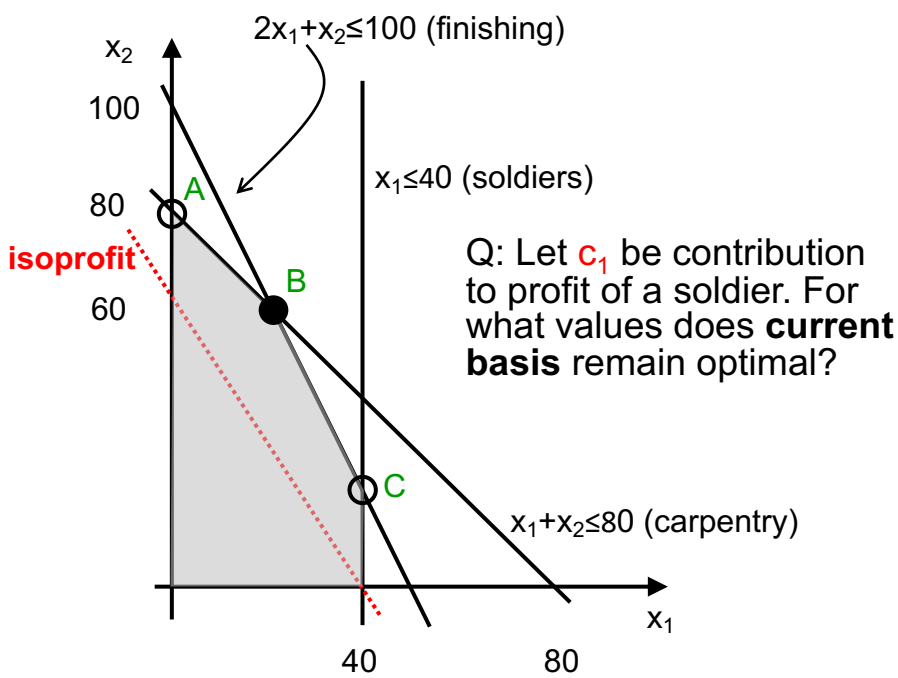
x_1 = number of soldiers produced
x_2 = number of trains produced

$$
\begin{array}{llll}
\max & z = 3x_1 + 2x_2 & & \\
\text{s.t.} & 2x_1 + x_2 & \leq 100 & \text{finishing} \\
 & x_1 + x_2 & \leq 80 & \text{carpentry} \\
 & x_1 & \leq 40 & \text{soldier demand} \\
 & x_1, x_2 & \geq 0 &
\end{array}
$$



- How would optimal solution change if objective coefficient or RHS values change?

Sensitivity to Objective Coefficient



Q: Let c_1 be contribution to profit of a soldier. For what values does **current basis** remain optimal?

- Isoprofit line is:

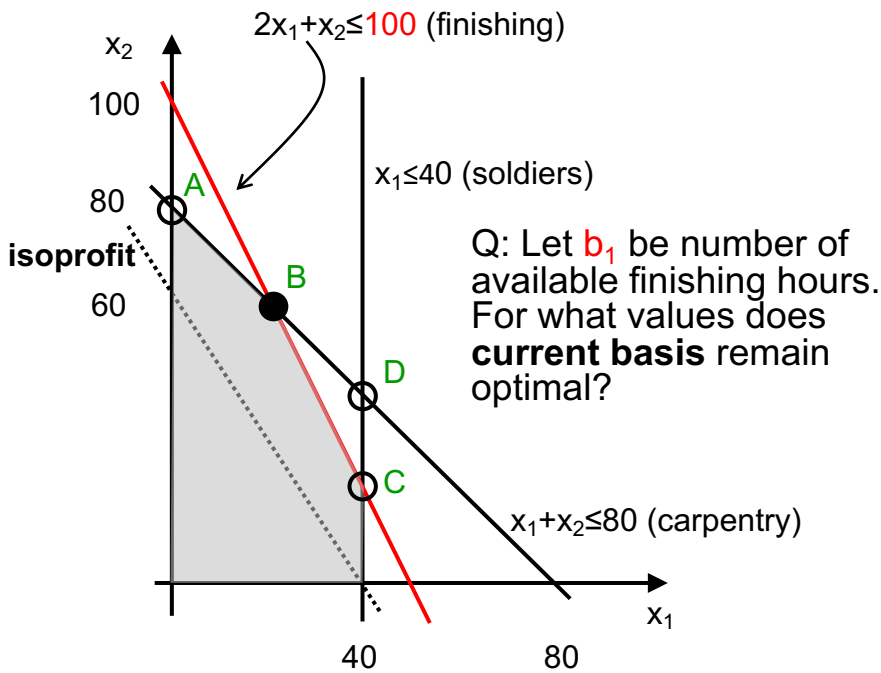
$$c_1 x_1 + 2x_2 = \text{constant}$$
$$x_2 = -\frac{c_1}{2} x_1 + \text{constant}/2$$
$$\Longrightarrow \text{slope is} - \frac{c_1}{2}$$

- Slope of carpentry constraint is -1
  - isoprofit lines "flatter" than this if $-c_1/2 > -1$, or $c_1 < 2$
  - New optimal solution would be at **A**
- Slope of finishing constraint is -2
  - isoprofit lines "steeper" than this if $-c_1/2 < -2$, or $c_1 > 4$.
  - New optimal solution would be at **C**

$\Longrightarrow$ Basis remains optimal for $2 \leq c_1 \leq 4$

- Would still manufacture 20 soldiers and 60 trains
- Of course profit changes! If $c_1 = 4$, profit will be $4(20)+2(60) = \$200$

Sensitivity to Right-hand side



- Current basis remains optimal as long as intersection of carpentry and finishing constraints remains feasible.
- Consider these two binding constraints:

$$2x_1 + x_2 = b_1$$
$$x_1 + x_2 = 80$$
$$\implies x_1 = b_1 - 80$$

- See that when $b_1 < 80$ then $x_1 < 0$ and basis infeasible. When $b_1 > 120$ then $x_1 > 40$ and basis infeasible.
  $\implies$ **Basis remains optimal for $80 \leq b_1 \leq 120$.**
- Within the range, both decision and objective value changes: if $b_1 = 100 + \epsilon$ then $x_1 = 20 + \epsilon$ and $x_2 = 60 - \epsilon$; $z = 180 + \epsilon$

  (Do you see why?)

Shadow prices

- **Definition.** The **shadow price** on the ith constraint is the amount by which objective value is improved if RHS b_i is increased by 1 (while current basis remains optimal.)

- For example, we know that if $b_1 = 100 + \epsilon$, then $x_1 = 20 + \epsilon$, $x_2 = 60 - \epsilon$ and $z = 3x_1 + 2x_2 = 180 + \epsilon$

- **Shadow price** of the "finishing" constraint is \$1; it is \$0 for 3rd (non-binding) constraint on demand of soldiers.

•<u>We will see that</u>:

shadow price =

optimal dual value corresponding to constraint =

reduced cost on slack var in optimal primal tableau

Second Approach: Using AMPL Sensitivity Report

Example 2: Furniture

- Make desks, tables and chairs
- Profit of \$60, \$30 and \$20 respectively
- Have 48' lumber, 20 finishing hours, 8 carpentry hours. Goal: maximize profit

Amount of each resource needed to make each type of furniture

	Desk	Table	Chair
Lumber	8'	6'	1'
Finishing	4 hrs	2 hrs	1.5 hrs
Carpentry	2 hrs	1.5 hrs	0.5 hrs

(Use LP, assume solution will be integral)

Example 2: Furniture

x_1 desks; x_2 tables; x_3 chairs

$$
\begin{array}{lll}
\max & z = 60x_1 + 30x_2 + 20x_3 & \\
\text{s.t.} & 8x_1 + 6x_2 + x_3 + x_4 = 48 & \text{lumber} \\
& 4x_1 + 2x_2 + 1.5x_3 + x_5 = 20 & \text{finishing} \\
& 2x_1 + 1.5x_2 + 0.5x_3 + x_6 = 8 & \text{carpentry} \\
& x_1, \ldots, x_6 \geq 0 &
\end{array}
$$

$$
\boxed{\begin{array}{l}
B = \{4, 3, 1\} \\
x^* = (2, 0, 8, 24, 0, 0) \\
z = 280
\end{array}}
$$

Optimal (primal) tableau:

$$
\begin{array}{lrrrrrrl}
z & & +5x_2 & & & + 10x_5 & + 10x_6 = & 280 \\
 & & - 2x_2 & & + x_4 & + 2x_5 & - 8x_6 = & 24 \\
 & & - 2x_2 & + x_3 & & + 2x_5 & - 4x_6 = & 8 \\
 & x_1 & + 1.25x_2 & & & -0.5x_5 & +1.5x_6 = & 2
\end{array}
$$

AMPL: Step 1 (furniture.mod)

```
# AMPL script for the Furniture model.
set PROD := 1..6;

# Decision variables (production program)
var X {j in PROD} >= 0;

# Objective function
maximize Obj: 60*X[1] + 30*X[2] + 20*X[3];

# Constraints
subject to Lumber: 8*X[1]+6*X[2]+1*X[3]+1*X[4]=48;
subject to Finishing: 4*X[1]+2*X[2]+1.5*X[3]+1*X[5]=20;
subject to Carpentry: 2*X[1]+1.5*X[2]+0.5*X[3]+1*X[6]=8;

end;
```

Data hard coded in the model file here. Not a good practice for large LP instances.

AMPL: Step 2 (furniture.run)

```
reset;
reset data;

model furniture.mod;

option solver './cplex';
option cplex_options 'sensitivity primalopt';
option presolve 0;
solve;

display X > furniture.sens;
display _varname, _var.rc, _var.down, _var.current,
  _var.up > furniture.sens;
display _conname, _con.dual, _con.down, _con.current,
  _con.up > furniture.sens;
```

Prevents solver using algebraic manipulation to simplify and remove variables, constraints before solving

AMPL: Step 3

```
ampl: include furniture.run;
```

--> Now look at furniture.sens file

```
X [*] :=
1   2
2   0
3   8
4  24
5   0
6   0
;
```

Note: reduced cost (.rc) in AMPL is the negation of our reduced cost.

reduced cost — *changes in obj fcn coeff's*

```
:  _varname  _var.rc  _var.down  _var.current  _var.up    :=
1   'X[1]'       0        56          60         80
2   'X[2]'      -5     -1e+20         30         35
3   'X[3]'       0        15          20         22.5
4   'X[4]'       0        -5           0          1.25
5   'X[5]'     -10     -1e+20          0         10
6   'X[6]'     -10     -1e+20          0         10
;
```

USEFUL VALUES

slack vars, less interesting

dual (shadow price) — *changes in RHS values*

```
:    _conname    _con.dual  _con.down  _con.current  _con.up    :=
1   Lumber          0         24            48        1e+20
2   Finishing      10         16            20           24
3   Carpentry      10         6.66667        8           10
;
```

USEFUL VALUES

Questions (answer using AMPL's report)

•If desks (x_1) were selling for $10 more per desk, how much more profit would we make?

->

•What if desks sold for $30 more per desk?

->

•If we have 2 fewer finishing hours, how would the profits change?

->

•If we have 3 more feet of lumber, how would the profits change?

->

Interpreting shadow prices

reduced cost *changes in obj fcn coeff's*

:	_varname	_var.rc	_var.down	_var.current	_var.up	:=
1	'X[1]'	0	56	60	80	
2	'X[2]'	-5	-1e+20	30	35	
3	'X[3]'	0	15	20	22.5	
4	'X[4]'	0	-5	0	1.25	
5	'X[5]'	-10	-1e+20	0	10	
6	'X[6]'	-10	-1e+20	0	10	

dual (shadow price) *changes in RHS values*

:	_conname	_con.dual	_con.down	_con.current	_con.up	:=
1	Lumber	0	24	48	1e+20	
2	Finishing	10	16	20	24	
3	Carpentry	10	6.66667	8	10	

- Q: How much would you be willing to pay for one additional carpentry hour?
- A: Since profits $60, $30, $20 incorporate costs of material, you'd pay up to $10 more than regular cost (since profit = $10)

Careful: Multiple changes at once

•The valid range of changes is only applicable when a **single** change is made.

•What if we want to consider sensitivity to multiple changes at once?

•**Case 1**: If changes are only in obj coefficients on **non-basic variables** and the RHS for **non-binding constraints**, simultaneous change within ranges ok.

•**Case 2**: Otherwise, need to use the **100% rule**:

- Defines the allowable simultaneous changes to the obj coeffs of basic variables.
- Defines the allowable simultaneous changes to RHS values of binding constraints.

The 100% rule

- Changes in objective coefficients

```
X [*] :=
1   2
2   0
3   8
4  24
5   0
6   0
;
```

Note: reduced cost (.rc) in AMPL is the negation of our reduced cost.

```
: _varname _var.rc _var.down _var.current _var.up    :=
1   'X[1]'      0        56        60        80
2   'X[2]'     -5    -1e+20        30        35
3   'X[3]'      0        15        20        22.5
4   'X[4]'      0        -5         0         1.25
5   'X[5]'    -10    -1e+20         0        10
6   'X[6]'    -10    -1e+20         0        10
;
```

<- to 70

<- to 18

OK

```
:    _conname    _con.dual  _con.down _con.current _con.up   :=
1   Lumber           0       24              48       1e+20
2   Finishing       10       16              20          24
3   Carpentry       10       6.66667          8          10
;
```

```
X [*] :=
1   2
2   0
3   8
4  24
5   0
6   0
;
```

Note: reduced cost (.rc) in AMPL is the negation of our reduced cost.

```
: _varname _var.rc _var.down _var.current _var.up    :=
1   'X[1]'      0        56        60        80
2   'X[2]'     -5    -1e+20        30        35
3   'X[3]'      0        15        20        22.5
4   'X[4]'      0        -5         0         1.25
5   'X[5]'    -10    -1e+20         0        10
6   'X[6]'    -10    -1e+20         0        10
;
```

<- to 70

<- to 22

Not OK

```
:    _conname    _con.dual  _con.down _con.current _con.up   :=
1   Lumber           0       24              48       1e+20
2   Finishing       10       16              20          24
3   Carpentry       10       6.66667          8          10
;
```

- **100% rule for objective coefficient changes**
  - if change is made on c_j to one or more basic variables then need $\sum_j r_j \leq 1$ where r_j is the **ratio** change for variable x_j with respect to its valid range.

- E.g., in furniture example, if profit of desks $70 and chairs $18 the current solution remains optimal because $r_1 = |70 - 60|/20 = 0.5, r_3 = |18 - 20|/5 = 0.4$, $\sum r_j = 0.9 < 1$.

The 100% rule

- Changes in RHS values

```
X [*] :=
1   2
2   0
3   8
4  24
5   0
6   0
;
```

Note: reduced cost (.rc) in AMPL is the negation of our reduced cost.

```
: _varname _var.rc _var.down _var.current _var.up    :=
1   'X[1]'      0        56        60       80
2   'X[2]'     -5    -1e+20        30       35
3   'X[3]'      0        15        20       22.5
4   'X[4]'      0        -5         0        1.25
5   'X[5]'    -10    -1e+20         0       10
6   'X[6]'    -10    -1e+20         0       10
;

:    _conname    _con.dual  _con.down _con.current _con.up    :=
1   Lumber           0       24              48       1e+20
2   Finishing       10       16              20          24  <- to 22
3   Carpentry       10       6.66667          8          10  <- to 9
;
```

OK

```
X [*] :=
1   2
2   0
3   8
4  24
5   0
6   0
;
```

Note: reduced cost (.rc) in AMPL is the negation of our reduced cost.

```
: _varname _var.rc _var.down _var.current _var.up    :=
1   'X[1]'      0        56        60       80
2   'X[2]'     -5    -1e+20        30       35
3   'X[3]'      0        15        20       22.5
4   'X[4]'      0        -5         0        1.25
5   'X[5]'    -10    -1e+20         0       10
6   'X[6]'    -10    -1e+20         0       10
;

:    _conname    _con.dual  _con.down _con.current _con.up    :=
1   Lumber           0       24              48       1e+20
2   Finishing       10       16              20          24  <- to 17
3   Carpentry       10       6.66667          8          10  <- to 9
;
```

Not OK

- **100% rule for RHS changes**
  - if change is made on b_i for one or more binding constraints then need $\sum_i r_i \leq 1$ where r_i is the **ratio** change for RHS b_i.

- E.g., in furniture example, if have 22 finishing hours and 9 carpentry hours then $r_2 = |22 - 20|/4 = 0.5$, $r_3 = |9 - 8|/2 = 0.5$. OK, because $\sum r_i = 1$.

Third Approach: Use "Basis to Tableau" Eqns and Optimal Primal Tableau

ORIGINAL PROBLEM

$\updownarrow$

FINAL TABLEAU

Recall: Furniture example

x_1 desks; x_2 tables; x_3 chairs

$$
\begin{array}{lrl}
\max & z = 60x_1 + 30x_2 + 20x_3 & \\
\text{s.t.} & 8x_1 + 6x_2 + x_3 + x_4 & = 48 \\
 & 4x_1 + 2x_2 + 1.5x_3 + x_5 & = 20 \\
 & 2x_1 + 1.5x_2 + 0.5x_3 + x_6 & = 8 \\
 & x_1, \ldots, x_6 & \geq 0
\end{array}
$$

$$
\boxed{\begin{array}{l}
B = \{4, 3, 1\} \\
x^* = (2, 0, 8, 24, 0, 0) \\
z = 280
\end{array}}
$$

Optimal (primal) tableau:

$$
\begin{array}{lrrrrrrr}
z & & +5x_2 & & & + 10x_5 & + 10x_6 = & 280 \\
 & & - 2x_2 & & + x_4 & + 2x_5 & - 8x_6 = & 24 \\
 & & - 2x_2 & + x_3 & & + 2x_5 & - 4x_6 = & 8 \\
 & x_1 & + 1.25x_2 & & & -0.5x_5 & +1.5x_6 = & 2
\end{array}
$$

- Variables $x_1, \ldots, x_6$.
- An **optimal tableau** looks like:

$$\begin{array}{lllll} z & + 5x_2 & + 10x_5 & + 10x_6 = & 280 \\ & & & = & 24 \\ & & & = & 8 \\ & & & = & 2 \end{array}$$

. . . with any values in the constraint matrix that leave the basic variables isolated.

- In considering whether change in LP data will cause optimal basis to change, we determine how changes affect RHS and row 0 of optimal tableau
- **Need $\bar{b} \geq 0$ (for feasibility) and $\bar{c} \geq 0$ (for optimality)**

Review: Tableau from a Basis

- Given an optimal (primal) basis B, then:

  RHS: $\bar{\mathbf{b}} = \mathbf{A_B^{-1} b}$

  Dual solution: $y^T = c_B^T A_B^{-1}$ (last lecture)

  Objective value: $z = c_B^T A_B^{-1} b = y^T b$

  Nonbasic obj coeff: $\bar{c_{B'}}^T = (c_B^T A_B^{-1} A_{B'} - c_{B'}^T)$

  For nonbasic j, $\bar{\mathbf{c}}_\mathbf{j} = \mathbf{c_B^T A_B^{-1} A_j} - \mathbf{c_j} = \mathbf{y^T A_j} - \mathbf{c_j}$

- Immediate observations:
  (a) $\bar{c}_j = y_j$ for slack variable x_j since $c_j = 0$ and $A_j = e_j$ (i.e., the jth unit vector) ← can read optimal dual directly from optimal primal tableau
  (b) dual variable y_j is shadow price on RHS of constraint j (since $z = y^T b$) ← the optimal dual value gives "shadow price"

Aside: Reading A_B^{-1} from the final tableau

Let $\tilde{B}$ and $\tilde{B}'$ denote the isolated and non-isolated variables in the initial tableau.

We have

$$A_{\tilde{B}} x_{\tilde{B}} + A_{\tilde{B}'} x_{\tilde{B}'} = b, \tag{1}$$

with $I = x_{\tilde{B}}$ by arranging columns according to order of $\tilde{B}$.

To get to the tableau for the optimal basis B, we multiply (1) by A_B^{-1}, to obtain

$$\bar{A}x = A_B^{-1} A_{\tilde{B}} x_{\tilde{B}} + A_B^{-1} A_{\tilde{B}'} x_{\tilde{B}'} = A_B^{-1} b = \bar{b} \tag{2}$$

In particular, since $I = A_{\tilde{B}}$ this means we can read off A_B^{-1} from the coefficients $\bar{A}_{\tilde{B}}$ $(= A_B^{-1} A_{\tilde{B}} = A_B^{-1})$ in the final tableau that correspond to the columns of the initial isolated variables $\tilde{B}$.

Furniture example

x_1 desks; x_2 tables; x_3 chairs

$$\begin{array}{lll} \max & z = 60x_1 + 30x_2 + 20x_3 & \\ \text{s.t.} & 8x_1 + 6x_2 + x_3 + x_4 & = 48 \\ & 4x_1 + 2x_2 + 1.5x_3 + x_5 & = 20 \\ & 2x_1 + 1.5x_2 + 0.5x_3 + x_6 & = 8 \\ & x_1, \ldots, x_6 & \geq 0 \end{array}$$

$$\boxed{\begin{array}{l} B = \{4, 3, 1\} \\ x^* = (2, 0, 8, 24, 0, 0) \\ z = 280 \end{array}}$$

Optimal (primal) tableau:

dual values (and **shadow prices**) from reduced costs on slack vars

$$\begin{array}{lrrrrrrl} z & & +5x_2 & & & \boxed{+\ 10x_5} & \boxed{+\ 10x_6} & = 280 \\ & & -\ 2x_2 & & +\ x_4 & +\ 2x_5 & -\ 8x_6 & = 24 \\ & & -\ 2x_2 & +\ x_3 & & +\ 2x_5 & -\ 4x_6 & = 8 \\ & x_1 & +\ 1.25x_2 & & & -0.5x_5 & +1.5x_6 & = 2 \end{array}$$

Example Analyses

A. Changing objective function coefficient of a non-basic variable

B. Changing objective function coefficient of a basic variable

C. Changing the RHS

D. Changing the entries in column of a non-basic variable

E. Introducing a new activity (decision variable)

A. Changing objective coeff. of non-basic variable

- Consider x_2 (tables) and change in c_2
- $\bar{b} = A_B^{-1} b$, so RHS does not change while basis B is unchanged
- $\mathbf{\bar{c}_j = c_B^T A_B^{-1} A_j - c_j}$ for $j \neq 2$ is unchanged
- **Must check reduced cost on $\bar{c}_2$ remains non-negative**
- Require $\bar{c}_2 = y^T A_2 - c_2 \geq 0$ (notice y^T constant) -> gives $c_2 \leq 35$
- Can also read from optimal primal tableau

Need $\bar{b} \geq 0$ (for feasibility) and $\bar{c} \geq 0$ (for optimality)

Furniture example

x_1 desks; x_2 tables; x_3 chairs

$$
\begin{array}{lll}
\max & z = 60x_1 + 30x_2 + 20x_3 & \\
\text{s.t.} & 8x_1 + 6x_2 + x_3 + x_4 & = 48 \\
& 4x_1 + 2x_2 + 1.5x_3 + x_5 & = 20 \\
& 2x_1 + 1.5x_2 + 0.5x_3 + x_6 & = 8 \\
& x_1, \ldots, x_6 & \geq 0
\end{array}
$$

$B = \{4, 3, 1\}$
$x^* = (2, 0, 8, 24, 0, 0)$
$z = 280$

Optimal (primal) tableau:

reduced cost

$$
\begin{array}{lrrrrrrl}
z & \boxed{+5x_2} & & & + 10x_5 & + 10x_6 = & 280 \\
 & - 2x_2 & & + x_4 & + 2x_5 & - 8x_6 = & 24 \\
 & - 2x_2 & + x_3 & & + 2x_5 & - 4x_6 = & 8 \\
x_1 & + 1.25x_2 & & & -0.5x_5 & +1.5x_6 = & 2
\end{array}
$$

Check against AMPL

Note: reduced cost (.rc) in AMPL is the negation of our reduced cost.

```
X [*] :=
1   2
2   0
3   8
4  24
5   0
6   0
;

: _varname _var.rc _var.down _var.current _var.up    :=
1   'X[1]'      0        56        60       80
2   'X[2]'     -5    -1e+20        30       35
3   'X[3]'      0        15        20       22.5
4   'X[4]'      0        -5         0        1.25
5   'X[5]'    -10    -1e+20         0       10
6   'X[6]'    -10    -1e+20         0       10
;

:    _conname    _con.dual  _con.down _con.current _con.up   :=
1   Lumber           0       24             48       1e+20
2   Finishing       10       16             20          24
3   Carpentry       10       6.66667         8          10
;
```

B. Changing objective coeff. of basic variable

- x_1 and x_3 are basic variables
- RHS $\bar{b} = A_B^{-1} b$ unchanged while basis B is unchanged
- $\bar{c}_{B'}^T = c_B^T A_B^{-1} A_{B'} - c_{B'}^T$ may change for multiple variables (c_B changes)
- **Must check reduced cost on every non-basic variable remains non-negative**
- For example, suppose profit on x_1 (desks) increases by $\epsilon > 0$
- $c_B = (0, 20, 60 + \epsilon)$. See how y^T changes

$$\mathbf{y}^T = c_B^T A_B^{-1} = \begin{pmatrix} 0 & 20 & 60+\epsilon \end{pmatrix} \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix} = \begin{pmatrix} 0 & 10-0.5\epsilon & 10+1.5\epsilon \end{pmatrix}$$

Use $\bar{c}_j = y^T A_j - c_j$ to analyze new reduced cost coefficients
(y^T not constant here)

Need $\bar{b} \geq 0$ (for feasibility) and $\bar{c} \geq 0$ (for optimality)

$$\bar{c}_2 = y^T A_2 - c_2 = \begin{pmatrix} 0 & 10-0.5\epsilon & 10+1.5\epsilon \end{pmatrix} \begin{pmatrix} 6 \\ 2 \\ 1.5 \end{pmatrix} - 30 = 5 + 1.25\epsilon$$

$$\bar{c}_5 = y^T A_5 - c_5 = \begin{pmatrix} 0 & 10-0.5\epsilon & 10+1.5\epsilon \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 0 = 10 - 0.5\epsilon$$

$$\bar{c}_6 = y^T A_6 - c_6 = \begin{pmatrix} 0 & 10-0.5\epsilon & 10+1.5\epsilon \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - 0 = 10 + 1.5\epsilon$$

- For **non-negative reduced costs in row 0**, need:

$$\begin{aligned} 5 + 1.25\epsilon \geq 0 \quad &\Longrightarrow \epsilon \geq -4 \\ 10 - 0.5\epsilon \geq 0 \quad &\Longrightarrow \epsilon \leq 20 \\ 10 + 1.5\epsilon \geq 0 \quad &\Longrightarrow \epsilon \geq -20/3 \end{aligned}$$

- **Overall**: need $-4 \leq \epsilon \leq 20$. While $60 - 4 \leq c_1 \leq 60 + 20$, current basis remains optimal. x^* unchanged, but if $c_1 := 70$ then $2(10)$ additional profit and $z = 300$

Check against AMPL

Note: reduced cost (.rc) in AMPL is the negation of our reduced cost.

```
X [*] :=
1   2
2   0
3   8
4  24
5   0
6   0
;

: _varname _var.rc _var.down _var.current _var.up    :=
1   'X[1]'      0        56        60       80
2   'X[2]'     -5    -1e+20        30       35
3   'X[3]'      0        15        20       22.5
4   'X[4]'      0        -5         0        1.25
5   'X[5]'    -10    -1e+20         0       10
6   'X[6]'    -10    -1e+20         0       10
;

:    _conname    _con.dual  _con.down _con.current _con.up   :=
1   Lumber           0       24              48       1e+20
2   Finishing       10       16              20          24
3   Carpentry       10       6.66667          8          10
;
```

C. Changing the RHS

- $\bar{c}_j = c_B^T A_B^{-1} A_j - c_j$ unchanged while basis B is unchanged
- But $\bar{b} = A_B^{-1} b$ changes. **Check $\bar{b} \geq 0$ to keep feasibility**
- Consider $b_2 = 20 + \epsilon$ in furniture example
- $\bar{b} = \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix} \begin{pmatrix} 48 \\ 20+\epsilon \\ 8 \end{pmatrix} = \begin{pmatrix} 24+2\epsilon \\ 8+2\epsilon \\ 2-0.5\epsilon \end{pmatrix}$ (*)
- Need:
  $24 + 2\epsilon \geq 0 \implies \epsilon \geq -12$
  $8 + 2\epsilon \geq 0 \implies \epsilon \geq -4$
  $2 - 0.5\epsilon \geq 0 \implies \epsilon \leq 4$
- Overall, need $-4 \leq \epsilon \leq 4$, and $16 \leq b_2 \leq 24$
- Effect on decision variables given by (*). Effect on objective value is

$$z = y^T b = \begin{pmatrix} 0 & 10 & 10 \end{pmatrix} \begin{pmatrix} 48 \\ 20+\epsilon \\ 8 \end{pmatrix} = 280 + 10\epsilon$$

(y^T from opt. tableau, constant)

Need $\bar{b} \geq 0$ (for feasibility) and $\bar{c} \geq 0$ (for optimality)

Check against AMPL

Note: reduced cost (.rc) in AMPL is the negation of our reduced cost.

```
X [*] :=
1   2
2   0
3   8
4  24
5   0
6   0
;

: _varname _var.rc _var.down _var.current _var.up    :=
1   'X[1]'      0        56        60       80
2   'X[2]'     -5    -1e+20        30       35
3   'X[3]'      0        15        20       22.5
4   'X[4]'      0        -5         0        1.25
5   'X[5]'    -10    -1e+20         0       10
6   'X[6]'    -10    -1e+20         0       10
;

:    _conname    _con.dual  _con.down _con.current _con.up   :=
1   Lumber           0       24              48       1e+20
2   Finishing       10       16              20          24
3   Carpentry       10       6.66667          8          10
;
```

D. Changing column entries for non-basic variable

- Consider tables (x_2). Change to: profit of \$43, use lumber 5, finishing 2, carpentry 2.
- $\bar{b} = A_B^{-1} b$ unchanged while basis B is unchanged
- $\bar{c}_j = c_B^T A_B^{-1} A_j - c_j$; see that only change is to the reduced cost of non-basic variable x_2 (only place A_j and c_j appear)
- **"Pricing out" the activity. Check reduced cost remains ≥ 0** (note y^T constant).
- $\bar{c}_2 = \underbrace{\begin{pmatrix} 0 & 10 & 10 \end{pmatrix}}_{(y^T \text{ from opt. tableau, constant here})} \underbrace{\begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix}}_{\text{new } A_2} - \underbrace{c_2}_{\text{new } c_2} = 40 - 43 = -3$
- See that the current basis is no longer optimal because $\bar{c}_2 < 0$

Need $\bar{b} \geq 0$ (for feasibility) and $\bar{c} \geq 0$ (for optimality)

E. Introducing a new activity (decision variable)

- Footstools (x_7). Profit of \$15, use lumber 1, finishing 1, carpentry 1.
- $\bar{b} = A_B^{-1} b$ unchanged while basis B is unchanged
- $\bar{c}_j = c_B^T A_B^{-1} A_j - c_j$; unchanged for all existing variables.
- **Just need to "price out" the new activity. Check reduced cost remains ≥ 0** (again, y^T constant).
- $\bar{c}_7 = \begin{pmatrix} 0 & 10 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - c_7 = 20 - 15 = 5$

  (y^T from opt. tableau, constant here)
- Current basis remains optimal because $\bar{c}_7$ >= 0

Need $\bar{b} \geq 0$ (for feasibility) and $\bar{c} \geq 0$ (for optimality)

Review: Sensitivity analysis with "basis to tableau" equations

Change	**Effect on optimal solution**	**Current basis still optimal if:**
Non-basic objective function coefficient c_j	Reduced cost $\bar{c}_j$ is changed	Need reduced cost $\bar{c}_j \geq 0$
Basic objective function coefficient c_j	All reduced costs may change, obj. value changed	Need reduced cost $\bar{c}_i \geq 0$ for all $i \in B'$
RHS b_i of a constraint	RHS in opt. tableau, optimal soln, and obj value	Need RHS $\bar{b}_i \geq 0$ on each constraint
Changing column entries for a non-basic variable x_j or adding a new variable x_j	Changes reduced cost on non-basic variable, or introduces new reduced cost.	Reduced cost $\bar{c}_j \geq 0$

Need $\bar{b} \geq 0$ (for feasibility) and $\bar{c} \geq 0$ (for optimality)

Summary: LP Sensitivity Analysis

- Sensitivity analysis provides an understanding of the robustness of an LP solution
- Important that **optimal basis does not change**:
  - Reduced costs remain non-negative, RHS values in optimal tableau remain non-negative
- Different approaches include:
  - Geometric arguments
  - AMPL's sensitivity report
  - "Basis to tableau" equations