Lesson Plan: Sensitivity

• Explore effect of changes in obj coefficients, and constraints on the optimal solution
  – see a connection to duality

• Approaches:
  – Geometric intuition
  – Use AMPL to get sensitivity information
  – Use “basis to tableau” eqns

Jensen & Bard: 4.1
Sensitivity Analysis

• What happens if the data change slightly?
  – E.g., a change in right-hand side value
  – E.g., a change in objective coefficient
  – E.g., a new decision variable
  – E.g., modifying the entries in the column corresponding to a variable

• Important to understand the robustness of a solution.

First Approach: Graphical Intuition
Example 1: Wooden toys

- Make toy soldiers and toy trains
- $3$ profit per soldier, $2$ profit per train
- Skilled labor of two types (carpentry and finishing)
  - Soldier: 2 hours finishing, 1 hour carpentry
  - Train: 1 hour finishing, 1 hour carpentry
  - 100 finishing hours, 80 carpentry hours each week
  - Unlimited demand for trains, at most 40 soldiers purchased each week
- Goal: maximize profit
- (Use LP, assume solution will be integral)

\[
\begin{align*}
\text{max } z &= 3x_1 + 2x_2 \\
\text{s.t. } & 2x_1 + x_2 \leq 100 & \text{finishing} \\
& x_1 + x_2 \leq 80 & \text{carpentry} \\
& x_1 \leq 40 & \text{soldier demand} \\
& x_1, x_2 \geq 0
\end{align*}
\]

\(x_1\) = number of soldiers produced
\(x_2\) = number of trains produced
How would optimal solution change if objective coefficient or RHS values change?
Sensitivity to Objective Coefficient

Q: Let $c_1$ be contribution to profit of a soldier. For what values does current basis remain optimal?
• Isoprofit line is:
  \[ c_1 x_1 + 2x_2 = \text{constant} \]
  \[ x_2 = -\frac{c_1}{2} x_1 + \text{constant}/2 \]
  \[ \implies \text{slope is} \ -\frac{c_1}{2} \]

• Slope of carpentry constraint is -1
  – isoprofit lines “flatter” than this if \(-c_1/2 > -1\), or \(c_1 < 2\)
  – New optimal solution would be at A

• Slope of finishing constraint is -2
  – isoprofit lines “steeper” than this if \(-c_1/2 < -2\), or \(c_1 > 4\).
  – New optimal solution would be at C

\[ \implies \text{Basis remains optimal for} \ 2 \leq c_1 \leq 4 \]

• Would still manufacture 20 soldiers and 60 trains

• Of course profit changes! If \(c_1 = 4\), profit will be 4(20) + 2(60) = $200

Sensitivity to Right-hand side
• Current basis remains optimal as long as intersection of carpentry and finishing constraints remains feasible.

• Consider these two binding constraints:
  \[2x_1 + x_2 = b_1\]
  \[x_1 + x_2 = 80\]
  \[\implies x_1 = b_1 - 80\]

• See that when \(b_1 < 80\) then \(x_1 < 0\) and basis infeasible. When \(b_1 > 120\) then \(x_1 > 40\) and basis infeasible.
  \(\implies\) **Basis remains optimal for** \(80 \leq b_1 \leq 120\).

• Within the range, decision and objective value changes: if \(b_1 = 100 + \epsilon\) then \(x_1 = 20 + \epsilon\) and \(x_2 = 60 - \epsilon\); \(z = 180 + \epsilon\)
  (Do you see why?)
Shadow prices

- **Definition.** The shadow price on the $i^{th}$ constraint is the amount by which objective value is improved if RHS $b_i$ is increased by 1 (while current basis remains optimal.)

- For example, we know that if $b_1 = 100 + \epsilon$, then $x_1 = 20 + \epsilon$, $x_2 = 60 - \epsilon$ and $z = 3x_1 + 2x_2 = 180 + \epsilon$

- **Shadow price** of the “finishing” constraint is $1; it is $0 for 3^{rd}$ (non-binding) constraint on demand of soldiers.

- **We will see that:**
  
  $$\text{shadow price} =$$
  
  $$\text{optimal dual value corresponding to constraint} =$$
  
  $$\text{reduced cost on slack var in optimal primal tableau}$$

Second Approach: Using AMPL

Sensitivity Report
Example 2: Furniture

- Make desks, tables and chairs
- Profit of $60, $30 and $20 respectively
- Have 48’ lumber, 20 finishing hours, 8 carpentry hours. Goal: maximize profit

Amount of each resource needed to make each type of furniture

<table>
<thead>
<tr>
<th></th>
<th>Desk</th>
<th>Table</th>
<th>Chair</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lumber</td>
<td>8’</td>
<td>6’</td>
<td>1’</td>
</tr>
<tr>
<td>Finishing</td>
<td>4 hrs</td>
<td>2 hrs</td>
<td>1.5 hrs</td>
</tr>
<tr>
<td>Carpentry</td>
<td>2 hrs</td>
<td>1.5 hrs</td>
<td>0.5 hrs</td>
</tr>
</tbody>
</table>

(Use LP, assume solution will be integral)

Example 2: Furniture

\[ x_1 \text{ desks; } x_2 \text{ tables; } x_3 \text{ chairs} \]
\[ \text{max } z = 60x_1 + 30x_2 + 20x_3 \]
\[ \text{s.t. } \begin{align*}
8x_1 + 6x_2 + x_3 + x_4 & = 48 & \text{lumber} \\
4x_1 + 2x_2 + 1.5x_3 + x_5 & = 20 & \text{finishing} \\
2x_1 + 1.5x_2 + 0.5x_3 + x_6 & = 8 & \text{carpentry} \\
x_1, \ldots, x_6 & \geq 0 & \\
\end{align*} \]

Optimal (primal) tableau:

\[ z \quad +5x_2 \quad +10x_5 \quad +10x_6 = 280 \]
\[ -2x_2 \quad +x_4 \quad +2x_5 \quad -8x_6 = 24 \]
\[ -2x_2 \quad +x_3 \quad +2x_5 \quad -4x_6 = 8 \]
\[ x_1 \quad +1.25x_2 \quad -0.5x_5 \quad +1.5x_6 = 2 \]
AMPL: Step 1 (furniture.mod)

# AMPL script for the Furniture model.
set PROD := 1..6;

# Decision variables (production program)
var X {j in PROD} >= 0;

# Objective function

# Constraints
end;

Data hard coded in the model file here. Not a good practice for large LP instances.

AMPL: Step 2 (furniture.run)

reset;
reset data;
model furniture.mod;

option solver './cplex';
option cplex_options 'sensitivity primalopt';
option presolve 0;
solve;

display X > furniture.sens;
display _varname, _var.rc, _var.down, _var.current, _var.up > furniture.sens;
display _conname, _con.dual, _con.down, _con.current, _con.up > furniture.sens;
AMPL: Step 3

```
AMPL: include furniture.run;
```

--> Now look at furniture.sens file

```
X [*] :=
1   2
2   0
3   8
4   24
5   0
6   0;

: _varname _var.rc _var.down _var.current _var.up :=
1   'X[1]'      0        56        60       80
2   'X[2]'      -5       -1e+20       30       35
3   'X[3]'      0        15        20       22.5
4   'X[4]'      0        -5        0        1.25
5   'X[5]'      -10      -1e+20       0        10
6   'X[6]'      -10      -1e+20       0        10;

: _conname   _con.dual  _con.down _con.current _con.up   :=
1   Lumber    0        24            48       1e+20
2   Finishing 10        16            20        24
3   Carpentry 10        6.66667       8        10
```

Note: reduced cost (rc) in AMPL is the negation of our reduced cost.

REDUCED COST

changes in obj fcn coeff’s

changes in RHS values

slack vars, less interesting

DUAL (SHADOW PRICE)

changes in RHS values

USEFUL VALUES
Questions (answer using AMPL’s report)

• If desks ($x_1$) were selling for $10 more per desk, how much more profit would we make?
  ->

• What if desks sold for $30 more per desk?
  ->

• If we have 2 fewer finishing hours, how would the profits change?
  ->

• If we have 3 more feet of lumber, how would the profits change?
  ->

Interpreting shadow prices

<table>
<thead>
<tr>
<th>varname</th>
<th>var.rc</th>
<th>var.down</th>
<th>var.current</th>
<th>var.up</th>
</tr>
</thead>
<tbody>
<tr>
<td>'X[1]'</td>
<td>0</td>
<td>56</td>
<td>60</td>
<td>80</td>
</tr>
<tr>
<td>'X[2]'</td>
<td>-5</td>
<td>-1e+20</td>
<td>30</td>
<td>35</td>
</tr>
<tr>
<td>'X[3]'</td>
<td>0</td>
<td>15</td>
<td>20</td>
<td>22.5</td>
</tr>
<tr>
<td>'X[4]'</td>
<td>0</td>
<td>-5</td>
<td>0</td>
<td>1.25</td>
</tr>
<tr>
<td>'X[5]'</td>
<td>-10</td>
<td>-1e+20</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>'X[6]'</td>
<td>-10</td>
<td>-1e+20</td>
<td>0</td>
<td>10</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>conname</th>
<th>con.dual</th>
<th>con.down</th>
<th>con.current</th>
<th>con.up</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lumber</td>
<td>0</td>
<td>24</td>
<td>48</td>
<td>1e+20</td>
</tr>
<tr>
<td>Finishing</td>
<td>10</td>
<td>16</td>
<td>20</td>
<td>24</td>
</tr>
<tr>
<td>Carpentry</td>
<td>10</td>
<td>6.66667</td>
<td>8</td>
<td>10</td>
</tr>
</tbody>
</table>

• Q: How much would you be willing to pay for one additional carpentry hour?
  • A: Since profits $60, $30, $20 incorporate costs of material, you’d pay up to $10 more than regular cost (since profit = $10)
Careful: Multiple changes at once

• The valid range of changes is only applicable when a single change is made.
• What if we want to consider sensitivity to multiple changes at once?
  • **Case 1**: If changes are only in obj coefficients on non-basic variables and the RHS for non-binding constraints, simultaneous change within ranges ok.
  • **Case 2**: Otherwise, need to use the **100% rule**:
    – Defines the allowable simultaneous changes to the obj coeffs of basic variables.
    – Defines the allowable simultaneous changes to RHS values of binding constraints.

The 100% rule

• Changes in objective coefficients
X [*] :=
1 2
2 0
3 8
4 24
5 0
6 0
;

: _varname _var.rc _var.down _var.current _var.up :=
1 'X[1]' 0 56 60 80 <- to 70
2 'X[2]' -5 -1e+20 30 35
3 'X[3]' 0 15 20 22.5 <- to 18
4 'X[4]' 0 -5 0 1.25
5 'X[5]' -10 -1e+20 0 10
6 'X[6]' -10 -1e+20 0 10 Not OK
;

: _conname _con.dual _con.down _con.current _con.up :=
1 Lumber 0 24 48 1e+20
2 Finishing 10 16 20 24
3 Carpentry 10 6.66667 8 10
;

Note: reduced cost (rc) in AMPL is the negation of our reduced cost.
• **100% rule for objective coefficient changes**
  
  – if change is made on $c_j$ to one or more basic variables then need $\sum_j r_j \leq 1$ where $r_j$ is the ratio change for variable $x_j$ with respect to its valid range.

• E.g., in furniture example, if profit of desks $70$ and chairs $18$ the current solution remains optimal because $r_1 = \frac{|70 - 60|}{20} = 0.5$, $r_3 = \frac{|18 - 20|}{5} = 0.4$, $\sum r_j = 0.9 < 1$.

### The 100% rule

• Changes in RHS values
\[
X [\ast] :=
\begin{align*}
1 & 2 \\
2 & 0 \\
3 & 8 \\
4 & 24 \\
5 & 0 \\
6 & 0 \\
\end{align*}
\]

Note: reduced cost (.rc) in AMPL is the negation of our reduced cost.

\[
; \quad \text{varname var.rc var.down var.current var.up} :=
\begin{align*}
1 & 'X[1]' 0 56 60 80 \\
2 & 'X[2]' -5 -1e+20 30 35 \\
3 & 'X[3]' 0 15 20 22.5 \\
4 & 'X[4]' 0 -5 0 1.25 \\
5 & 'X[5]' -10 -1e+20 0 10 \\
6 & 'X[6]' -10 -1e+20 0 10 \\
\end{align*}
\]

\[
; \quad \text{conname con.dual con.down con.current con.up} :=
\begin{align*}
1 & \text{Lumber} 0 24 48 1e+20 \\
2 & \text{Finishing} 10 16 20 24 < \text{to} 22 \\
3 & \text{Carpentry} 10 6.66667 8 10 < \text{to} 9 \\
\end{align*}
\]

\[
\text{Not OK}
\]
• 100% rule for RHS changes
  – if change is made on $b_i$ for one or more binding constraints then need $\sum r_i \leq 1$ where $r_i$ is the ratio change for RHS $b_i$.

• E.g., in furniture example, if have 22 finishing hours and 9 carpentry hours then $r_2 = |22 - 20|/4 = 0.5$, $r_3 = |9 - 8|/2 = 0.5$. OK, because $\sum r_i = 1$.

Third Approach: Use “Basis to Tableau” Eqns and Optimal Primal Tableau
Recall: Furniture example

\(x_1\) desks; \(x_2\) tables; \(x_3\) chairs

\[
\begin{align*}
\text{max } & \quad z = 60x_1 + 30x_2 + 20x_3 \\
\text{s.t. } & \quad 8x_1 + 6x_2 + x_3 + x_4 = 48 \\
& \quad 4x_1 + 2x_2 + 1.5x_3 + x_5 = 20 \\
& \quad 2x_1 + 1.5x_2 + 0.5x_3 + x_6 = 8 \\
& \quad x_1, \ldots, x_6 \geq 0 \\
\end{align*}
\]

\(B = \{4, 3, 1\}\)
\(x^* = (2, 0, 8, 24, 0, 0)\)
\(z = 280\)

Optimal (primal) tableau:

\[
\begin{array}{cccccc}
z & +5x_2 & +10x_5 & +10x_6 &= & 280 \\
-2x_2 & +x_4 & +2x_5 & -8x_6 &= & 24 \\
-2x_2 & +x_3 & +2x_5 & -4x_6 &= & 8 \\
x_1 & +1.25x_2 & -0.5x_5 & +1.5x_6 &= & 2 \\
\end{array}
\]
• Variables $x_1, \ldots, x_6$.

• An **optimal tableau** looks like:

\[
\begin{align*}
    z + 5x_2 & + 10x_5 + 10x_6 = 280 \\
    & = 24 \\
    & = 8 \\
    & = 2
\end{align*}
\]

... with any values in the constraint matrix that leave the basic variables isolated.

• In considering whether change in LP data will cause optimal basis to change, we determine how changes affect RHS and row 0 of optimal tableau

• **Need** $\bar{b} \geq 0$ (for feasibility) and $\bar{c} \geq 0$ (for optimality)

---

**Review: Tableau from a Basis**

• Given an optimal (primal) basis $B$, then:

  **RHS:** $\bar{b} = A^{-1}_B b$

  **Dual solution:** $y^T = c^T_B A^{-1}_B$  \( \text{(last lecture)} \)

  **Objective value:** $z = c^T_B A^{-1}_B b = y^T b$

  **Nonbasic obj coeff:** $c_B^{-T} = (c^T_B A^{-1}_B A_B - c^T_B)$

  For nonbasic $j$, $\bar{c}_j = c^T_B A^{-1}_B A_B - c_j = y^T A_j - c_j$

• **Immediate observations:**

  (a) $\bar{c}_j = y_j$ for slack variable $x_j$ since $c_j = 0$ and $A_j = e_j$  \( \leftrightarrow \) can read optimal dual directly from optimal primal tableau

  (b) dual variable $y_j$ is shadow price on RHS of constraint $j$ (since $z = y^T b$)  \( \leftrightarrow \) the optimal dual value gives "shadow price"
Aside: Reading $A_B^{-1}$ from the final tableau

Let $\tilde{B}$ and $\tilde{B}'$ denote the isolated and non-isolated variables in the initial tableau.

We have

$$A_{\tilde{B}}x_{\tilde{B}} + A_{\tilde{B}'}x_{\tilde{B}'} = b,$$

with $I = x_{\tilde{B}}$ by arranging columns according to order of $\tilde{B}$.

To get to the tableau for the optimal basis $B$, we multiply (1) by $A_B^{-1}$, to obtain

$$\tilde{A}x = A_B^{-1}A_{\tilde{B}}x_{\tilde{B}} + A_B^{-1}A_{\tilde{B}'}x_{\tilde{B}'} = A_B^{-1}b = \bar{b}$$

In particular, since $I = A_{\tilde{B}}$ this means we can read off $A_B^{-1}$ from the coefficients $A_{\tilde{B}} (= A_B^{-1}A_{\tilde{B}} = A_{\tilde{B}'}^{-1})$ in the final tableau that correspond to the columns of the initial isolated variables $\tilde{B}$.

---

**Furniture example**

$x_1$ desks; $x_2$ tables; $x_3$ chairs

$$\begin{align*}
\text{max} \quad z & = 60x_1 + 30x_2 + 20x_3 \\
\text{s.t.} \quad 8x_1 + 6x_2 + x_3 + x_4 & = 48 \\
& \quad 4x_1 + 2x_2 + 1.5x_3 + x_5 = 20 \\
& \quad 2x_1 + 1.5x_2 + 0.5x_3 + x_6 = 8 \\
& \quad x_1, \ldots, x_6 \geq 0
\end{align*}$$

**Optimal (primal) tableau:**

<table>
<thead>
<tr>
<th></th>
<th>$-5x_2$</th>
<th>$+10x_5$</th>
<th>$+10x_6$</th>
<th>$= 280$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$+2x_2$</td>
<td>$+x_4$</td>
<td>$+2x_5$</td>
<td>$-8x_6$</td>
</tr>
<tr>
<td></td>
<td>$-2x_2$</td>
<td>$+x_3$</td>
<td>$+2x_5$</td>
<td>$-4x_6$</td>
</tr>
<tr>
<td></td>
<td>$+1.25x_2$</td>
<td></td>
<td>$-0.5x_5$</td>
<td>$+1.5x_6$</td>
</tr>
</tbody>
</table>

$B = \{4, 3, 1\} \quad x^* = (2, 0, 8, 24, 0, 0) \quad z = 280$

Dual values (and shadow prices) from reduced costs on slack vars
Example Analyses

A. Changing objective function coefficient of a non-basic variable
B. Changing objective function coefficient of a basic variable
C. Changing the RHS
D. Changing the entries in column of a non-basic variable
E. Introducing a new activity (decision variable)

A. Changing objective coeff.
of non-basic variable

- Consider $x_2$ (tables) and change in $c_2$
- $\bar{b} = A_B^{-1}b$, so RHS does not change while basis B is unchanged
- $\bar{c}_j = c_B^T A_B^{-1} A_j - c_j$ for $j \neq 2$ is unchanged
- Must check reduced cost on $\bar{c}_2$ remains non-negative
- Require $\bar{c}_2 = y^T A_2 - c_2 \geq 0$ (notice $y^T$ constant) $\rightarrow$ gives $c_2 \leq 35$
- Can also read from optimal primal tableau

Need $\bar{b} \geq 0$ (for feasibility) and $\bar{c} \geq 0$ (for optimality)
Furniture example

\[ x_1 \] desks; \[ x_2 \] tables; \[ x_3 \] chairs

\[
\begin{align*}
\text{max} & \quad z = 60x_1 + 30x_2 + 20x_3 \\
\text{s.t.} & \quad 8x_1 + 6x_2 + x_3 + x_4 = 48 \\
& \quad 4x_1 + 2x_2 + 1.5x_3 + x_5 = 20 \\
& \quad 2x_1 + 1.5x_2 + 0.5x_3 + x_6 = 8 \\
& \quad x_1, \ldots, x_6 \geq 0
\end{align*}
\]

Optimal (primal) tableau:

\[
\begin{array}{cccccc}
& +5x_2 & \text{reduced cost} & +10x_5 & +10x_6 & = 280 \\
-2x_2 & +x_4 & +2x_5 & -8x_6 & = 24 \\
-2x_2 & +x_3 & +2x_5 & -4x_6 & = 8 \\
x_1 & +1.25x_2 & -0.5x_5 & +1.5x_6 & = 2
\end{array}
\]

\[
B = \{4,3,1\} \\
x^* = (2,0,8,24,0,0) \\
z = 280
\]

Check against AMPL

Note: reduced cost (.rc) in AMPL is the negation of our reduced cost.

\[
X [*] :=
1 2 \\
2 0 \\
3 8 \\
4 24 \\
5 0 \\
6 0
\]

\[
\begin{array}{ccccccc}
\text{varname} & \text{var.rc} & \text{var.down} & \text{var.current} & \text{var.up} & := \\
1 'X[1]' & 0 & 56 & 60 & 80 \\
2 'X[2]' & -5 & -1e20 & 30 & 35 \\
3 'X[3]' & 0 & 15 & 20 & 22.5 \\
4 'X[4]' & 0 & -5 & 0 & 1.25 \\
5 'X[5]' & -10 & -1e20 & 0 & 10 \\
6 'X[6]' & -10 & -1e20 & 0 & 10
\end{array}
\]

\[
\begin{array}{cccccc}
\text{conname} & \text{con.dual} & \text{con.down} & \text{con.current} & \text{con.up} & := \\
1 \text{Lumber} & 0 & 24 & 48 & 1e20 \\
2 \text{Finishing} & 10 & 16 & 20 & 24 \\
3 \text{Carpentry} & 10 & 6.66667 & 8 & 10
\end{array}
\]
B. Changing objective coeff. of basic variable

- $x_1$ and $x_3$ are basic variables
- RHS $\bar{b} = A_B^{-1}b$ unchanged while basis $B$ is unchanged
- $\bar{c}'_B = c_B A_B^{-1} A_B - c_B'$ may change for multiple variables ($c_B$ changes)
- Must check reduced cost on every non-basic variable remains non-negative
  - For example, suppose profit on $x_1$ (desks) increases by $\epsilon > 0$
  - $c_B = (0, 20, 60 + \epsilon)$. See how $y^T$ changes
  
  \[
  y^T = c_B^T A_B^{-1} = \begin{pmatrix} 0 & 20 & 60 + \epsilon \end{pmatrix} \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix} = \begin{pmatrix} 0 & 10 - 0.5\epsilon & 10 + 1.5\epsilon \end{pmatrix}
  \]
  
  Use $\bar{c}_j = y^T A_j - c_j$ to analyze new reduced cost coefficients
  ($y^T$ not constant here)

**Need $\bar{b} \geq 0$ (for feasibility) and $\bar{c} \geq 0$ (for optimality)**

\[
\bar{c}_2 = y^T A_2 - c_2 = \begin{pmatrix} 0 & 10 - 0.5\epsilon & 10 + 1.5\epsilon \end{pmatrix} \begin{pmatrix} 6 \\ 2 \\ 1.5 \end{pmatrix} = 5 + 1.25\epsilon
\]

\[
\bar{c}_5 = y^T A_5 - c_5 = \begin{pmatrix} 0 & 10 - 0.5\epsilon & 10 + 1.5\epsilon \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 10 - 0.5\epsilon
\]

\[
\bar{c}_6 = y^T A_6 - c_6 = \begin{pmatrix} 0 & 10 - 0.5\epsilon & 10 + 1.5\epsilon \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 10 + 1.5\epsilon
\]

- For non-negative reduced costs in row 0, need:
  
  \[
  \begin{align*}
  5 + 1.25\epsilon & \geq 0 \implies \epsilon \geq -4 \\
  10 - 0.5\epsilon & \geq 0 \implies \epsilon \leq 20 \\
  10 + 1.5\epsilon & \geq 0 \implies \epsilon \geq -20/3
  \end{align*}
  \]

- Overall: need $-4 \leq \epsilon \leq 20$. While $60 - 4 \leq c_1 \leq 60 + 20$, current basis remains optimal. $x^*$ unchanged, but if $c_1 : = 70$ then $2(10)$ additional profit and $z = 300$
\[ X \begin{bmatrix} [*] \end{bmatrix} := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 2 & 0 & 8 & 24 & 0 & 0 \end{bmatrix}; \]

Check against AMPL

Note: reduced cost \((rc)\) in AMPL is the negation of our reduced cost.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{varname} & \text{var.rc} & \text{var.down} & \text{var.current} & \text{var.up} \\
\hline
'X[1]' & 0 & 56 & 60 & 80 \\
'X[2]' & -5 & -1e+20 & 30 & 35 \\
'X[3]' & 0 & 15 & 20 & 22.5 \\
'X[4]' & 0 & -5 & 0 & 1.25 \\
'X[5]' & -10 & -1e+20 & 0 & 10 \\
'X[6]' & -10 & -1e+20 & 0 & 10 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{conname} & \text{con.dual} & \text{con.down} & \text{con.current} & \text{con.up} \\
\hline
\text{Lumber} & 0 & 24 & 48 & 1e+20 \\
\text{Finishing} & 10 & 16 & 20 & 24 \\
\text{Carpentry} & 10 & 6.66667 & 8 & 10 \\
\hline
\end{array}
\]

C. Changing the RHS

- \( \bar{c}_j = c^T_B A^{-1}_B A_j - c_j \) unchanged while basis \( B \) is unchanged
- But \( \bar{b} = A^{-1}_B b \) changes. Check \( \bar{b} \geq 0 \) to keep feasibility
- Consider \( b_2 = 20 + \epsilon \) in furniture example
  
  \[
  \bar{b} = \begin{bmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 48 \\ 20 + \epsilon \\ 8 \end{bmatrix} = \begin{bmatrix} 24 + 2\epsilon \\ 8 + 2\epsilon \\ 2 - 0.5\epsilon \end{bmatrix}
  \]

- Need:
  
  \[
  24 + 2\epsilon \geq 0 \implies \epsilon \geq -12 \\
  8 + 2\epsilon \geq 0 \implies \epsilon \geq -4 \\
  2 - 0.5\epsilon \geq 0 \implies \epsilon \leq 4
  \]

- Overall, need \(-4 \leq \epsilon \leq 4\), and \( 16 \leq b_2 \leq 24 \)
- Effect on decision variables given by \((*)\). Effect on objective value is
  
  \[
  z = y^T b = \begin{bmatrix} 0 & 10 & 10 \end{bmatrix} \begin{bmatrix} 48 \\ 20 + \epsilon \\ 8 \end{bmatrix} = 280 + 10\epsilon
  \]

\((y^T \text{ from opt. tableau, constant})\)

Need \( \bar{b} \geq 0 \) (for feasibility) and \( \bar{c} \geq 0 \) (for optimality)
Check against AMPL

Note: reduced cost (rc) in AMPL is the negation of our reduced cost.

D. Changing column entries for non-basic variable

- Consider tables (x2). Change to: profit of $43, use lumber 5, finishing 2, carpentry 2.

- \( \bar{b} = A_B^{-1}b \) unchanged while basis B is unchanged

- \( \bar{c}_j = c_B^T A_B^{-1} A_j - c_j \); see that only change is to the reduced cost of non-basic variable \( x_2 \) (only place \( A_i \) and \( c \) appear)

- "Pricing out" the activity. Check reduced cost remains \( \geq 0 \)
  
  (note \( y^T \) constant).

- \( \bar{c}_2 = \begin{pmatrix} 0 & 10 & 10 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \\ 2 \end{pmatrix} = 40 - 43 = -3 \)

- See that the current basis is no longer optimal because \( \bar{c}_2 < 0 \)

Need \( \bar{b} \geq 0 \) (for feasibility) and \( \bar{c} \geq 0 \) (for optimality)
E. Introducing a new activity (decision variable)

- Footstools \((x_f)\). Profit \(\$15\), use lumber 1, finishing 1, carpentry 1.
- \(\bar{b} = A_B^{-1}b\) unchanged while basis \(B\) is unchanged
- \(\bar{c}_j = c_B^TA_B^{-1}A_j - c_j\); unchanged for all existing variables.
- Just need to “price out” the new activity. Check reduced cost remains \(\geq 0\) (again, \(y^T\) constant).

\[
\bar{c}_7 = \begin{pmatrix} 0 & 10 & 10 \\ \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \end{pmatrix} - c_7 = 20 - 15 = 5
\]

- Current basis remains optimal because \(\bar{c}_7 \geq 0\)

Need \(\bar{b} \geq 0\) (for feasibility) and \(\bar{c} \geq 0\) (for optimality)

Review: Sensitivity analysis with “basis to tableau” equations

<table>
<thead>
<tr>
<th>Change</th>
<th>Effect on optimal solution</th>
<th>Current basis still optimal if:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-basic objective function coefficient (c_j)</td>
<td>Reduced cost (\bar{c}_j) is changed</td>
<td>Need reduced cost (\bar{c}_j \geq 0)</td>
</tr>
<tr>
<td>Basic objective function coefficient (c_j)</td>
<td>All reduced costs may change, obj. value changed</td>
<td>Need reduced cost (\bar{c}_i \geq 0) for all (i \in B')</td>
</tr>
<tr>
<td>RHS (b_i) of a constraint</td>
<td>RHS in opt. tableau, optimal soln, and obj value</td>
<td>Need RHS (\bar{b}_i \geq 0) on each constraint</td>
</tr>
<tr>
<td>Changing column entries for a non-basic variable (x_j) or adding a new variable (x_i)</td>
<td>Changes reduced cost on non-basic variable, or introduces new reduced cost.</td>
<td>Reduced cost (\bar{c}_j \geq 0)</td>
</tr>
</tbody>
</table>

Need \(\bar{b} \geq 0\) (for feasibility) and \(\bar{c} \geq 0\) (for optimality)
Summary: LP Sensitivity Analysis

• Sensitivity analysis provides an understanding of the robustness of an LP solution
• Important that **optimal basis does not change**:  
  – Reduced costs remain non-negative, RHS values in optimal tableau remain non-negative
• Different approaches include:  
  – Geometric arguments  
  – AMPL’s sensitivity report  
  – “Basis to tableau” equations