AM 121: Intro to Optimization Models and Methods

Lecture 7: Duality Theory

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Lesson Plan

- Certificate of optimality
- Duality for the standard equality form
- Weak and strong duality theorems

Jensen & Bard: 4.2 before Complementary Slackness
Terminology

- Standard equality form
  \[ \max c^T x \]
  s.t. \[ Ax = b \]
  \[ x \geq 0 \]
- Standard inequality form
  \[ \max c^T x \]
  s.t. \[ Ax \leq b \]
  \[ x \geq 0 \]
- Canonical form
  - Equality form with \( b \geq 0 \), and isolated variables

Duality

- Linear programs come in primal/dual pairs
- Why is duality useful?
  - it leads to a variation called the “dual simplex” method that has practical impact
  - it provides economic insight into problems
  - it suggests a method to perform “sensitivity analysis” on an LP (next lecture)
  - it provides a new way to reason about optimality and infeasibility
Establishing optimality

• max $x_1 - x_2 + 7x_3$
  s.t. $2x_1 - x_2 + x_3 = 1$
  $x_1 + x_2 + 2x_3 = 5$
  $x_1, x_2, x_3 \geq 0$

• Every feasible solution provides a lower bound of the optimal objective value

• Suppose find $x=(0,1,2)$, value 13. Optimal val $\geq 13$.

• But how far away is $x=(0,1,2)$ from optimality?

Establishing optimality

• max $x_1 - x_2 + 7x_3$
  s.t. $2x_1 - x_2 + x_3 = 1$ \hspace{1cm} (a)
  $x_1 + x_2 + 2x_3 = 5$ \hspace{1cm} (b)
  $x_1, x_2, x_3 \geq 0$

• Example row operations:
  $3x_1 + 2x_2 + 7x_3 = 16$

  $x_1 - x_2 + 7x_3 \leq 5x_1 + 2x_2 + 7x_3 = 16$

• Consider $3(a) + 2(b)$:
  $\frac{3(2x_1 - x_2 + x_3)}{8} = \frac{3(1)}{8}$
  $\frac{x_1 - x_2 + 7x_3}{8} \leq \frac{8x_1 - x_2 + 7x_3}{8} = 13$

• Conclude $x^*=(0,1,2)$ is optimal.
Proof of Optimality

• Convince friend that $x^*$ is optimal by presenting with “optimal constraint multipliers” $y^*=(3,2)$.

• Friend follows steps:
  – Verify $x^*$ is feasible
  – Verify objective value $x^* \leq$ upper bound implied by multipliers $y^*$

• These optimal multipliers $y^*$ provide a **certificate of optimality** for $x^*$

• The dual problem will find these optimal multipliers

• Adding constraints, **any** feasible solution $x$ must satisfy:

$$y_1(2x_1 - x_2 + x_3) + y_2(x_1 + x_2 + 2x_3) = y_1 + 5y_2$$

$$\Rightarrow (2y_1+y_2)x_1 + (-y_1 + y_2)x_2 + (y_1+2y_2)x_3 = y_1 + 5y_2$$

• For an upper bound, we need:

$$x_1-x_2+7x_3 \leq (2y_1+y_2)x_1 + (-y_1 + y_2)x_2 + (y_1+2y_2)x_3 = y_1 + 5y_2$$

• Since $x \geq 0$, we need $1 \leq (2y_1 + y_2)$; $-1 \leq (-y_1 + y_2)$; $7 \leq (y_1 + 2y_2)$.

• Can solve:

$$\min \; y_1 + 5y_2$$

s.t.

$$2y_1 + y_2 \geq 1$$

$$-y_1 + y_2 \geq -1$$

$$y_1 + 2y_2 \geq 7$$

• This is the **dual** for the original (**primal**) LP. We have argued that the optimal value of (D) $\geq$ optimal value of (P).

• We have $y^*=(3,2)$ feasible for (D) with value 13; and $x^*=(0,1,2)$ feasible for (P) with value 13.

• So, $y^*$ optimal for (D); $x^*$ optimal for (P).
Example: Primal and Dual form

• \( \text{max } x_1 - x_2 + 7x_3 \) \hspace{1cm} (P)
  s.t. \( 2x_1 - x_2 + x_3 = 1 \)
  \( x_1 + x_2 + 2x_3 = 5 \)
  \( x_1, x_2, x_3 \geq 0 \)

• \( \text{min } y_1 + 5y_2 \) \hspace{1cm} (D)
  s.t. \( 2y_1 + y_2 \geq 1 \)
  \( -y_1 + y_2 \geq -1 \)
  \( y_1 + 2y_2 \geq 7 \)

• Observe:
  – coefficients objective -> values RHS
  – #variables -> #constraints
  – #constraints -> #variables
  – dual matrix is transpose of primal matrix
  – equality constraints -> free variables

Dual for Standard Equality Form

• \( \text{max } c^T x \) \hspace{1cm} (P)
  s.t. \( Ax = b \)
  \( x \geq 0 \)

• \( \text{min } b^T y \) \hspace{1cm} (D)
  s.t. \( A^T y \geq c \)

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
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<tbody>
<tr>
<td>= constraints</td>
<td>free variables</td>
</tr>
<tr>
<td>non-negative variables</td>
<td>( \geq ) constraints</td>
</tr>
</tbody>
</table>
Example: Furniture problem (1/2)

- Make desks, tables and chairs.
- Have 48 lumber hours, 20 finishing hours, 8 carpentry hours available.
- Desk profit $60, table $30 and chair $20.

<table>
<thead>
<tr>
<th></th>
<th>Desk</th>
<th>Table</th>
<th>Chair</th>
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<tbody>
<tr>
<td>Lumber</td>
<td>8</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>Finishing</td>
<td>4</td>
<td>2</td>
<td>1.5</td>
</tr>
<tr>
<td>Carpentry</td>
<td>2</td>
<td>1.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Amount of each resource needed to make each type of furniture

Example: Furniture problem (2/2)

- max \( z = 60x_1 + 30x_2 + 20x_3 \)
  s.t. \( 8x_1 + 6x_2 + x_3 + x_4 = 48 \)
  \( 4x_1 + 2x_2 + 1.5x_3 + x_5 = 20 \)
  \( 2x_1 + 1.5x_2 + 0.5x_3 + x_6 = 8 \)
  \( x_1, \ldots, x_6 \geq 0 \)

\( x^*=(2,0,8,24,0,0), \ z=280 \)

- min \( z = 48 y_1 + 20y_2 + 8y_3 \)
  s.t. \( 8y_1 + 4y_2 + 2y_3 \geq 60 \)
  \( 6y_1 + 2y_2 + 1.5y_3 \geq 30 \)
  \( y_1 + 1.5y_2 + 0.5y_3 \geq 20 \)
  \( y_1 \geq 0 \)
  \( y_2 \geq 0 \)
  \( y_3 \geq 0 \)

\( y^*=(0,10,10), \ z=280 \)
Economic interpretation

- Optimal multiplier $y_i$ corresponding to primal constraint $i$ corresponds to change in primal objective if primal RHS $b_i$ increased by 1.
- Dual var’s $y$ also called “shadow prices”.
- In the furniture example, represent the value of incremental resources
  - Lumber = $0$
  - Finishing = $10$
  - Carpentry = $10$

Additional supply of lumber not useful (it is already in surplus in optimal sol!)

Duality for General LPs

**Primal**
- $\max c^Tx + d^Tw$
  - s.t. $Ax + Ew = b$  \((y_i)\)
  - $Fx + Gw \leq e$  \((u_j)\)
  - $x \geq 0$, $w$ free

**Dual**
- $\min b^Ty + e^Tu$
  - s.t. $A^Ty + F^Tu \geq c$  \((x_k)\)
  - $E^Ty + G^Tu = d$  \((w_l)\)
  - $y$ free, $u \geq 0$

**Observations**
- \((AE \Rightarrow AT FT \ F G) \Rightarrow ET GT)\):
  - obj coeff. $\Rightarrow$ RHS coeff.
  - Equalities $\Rightarrow$ free vars
  - Inequal $\Rightarrow$ nonneg vars
  - nonneg vars $\Rightarrow$ equalities
  - free vars $\Rightarrow$ inequal
Example

• max \( x_1 - x_2 \)
  s.t. \(-3x_1 + x_2 = 2\) \( y_1 \) free
  \( 2x_1 - x_2 \leq -1 \) \( y_2 \) nonnegative
  \( x_1 \geq 0, x_2 \) free

• min \( 2y_1 - y_2 \)
  s.t. \(-3y_1 + 2y_2 \geq 1\) \( x_1 \) nonnegative
  \( y_1 - y_2 = -1 \) \( x_2 \) free
  \( y_1 \) free, \( y_2 \geq 0 \)

Standard inequality form:
Special case

• max \( c^T x \)
  s.t. \( A x \leq b \) \((y_i \) non-negative)\)
  \( x \geq 0 \)

• min \( b^T y \)
  s.t. \( A^T y \geq c \) \((x_j \) non-negative)\)
  \( y \geq 0 \)

Obtain this by setting \{d,A,E,G\} to zero
Weak Duality Theorem

Weak duality theorem: objective value of any feasible primal solution is weakly less than objective value of any feasible dual solution.

Weak duality theorem of LP

• **Theorem.** The (primal) objective value of any feasible primal solution weakly less than the (dual) objective value of any feasible dual solution.

• **Proof.** (only prove for the standard equality form here)

\[
\begin{align*}
\text{max} \quad & c^T x & \quad \text{(P)} \\
\text{s.t.} \quad & Ax = b \\
& x \geq 0 \\
\text{min} \quad & b^T y & \quad \text{(D)} \\
\text{s.t.} \quad & A^T y \geq c
\end{align*}
\]

Consider \( x \) feasible for (P) and \( y \) feasible for (D).
\[
c^T x \leq (A^T y)^T x = (y^T A)x = y^T Ax = y^T b = b^T y
\]
• **Corollary.** If a primal LP is unbounded then its dual problem is infeasible.

• **Proof.** Suppose (D) has a feasible solution $y$. Then by weak duality, primal optimal value $\leq b^T y$, and contradicts with unbounded primal.

• **Corollary.** If a dual LP is unbounded then its primal problem is infeasible.

• **Proof.** Suppose (P) has a feasible solution $x$. Then by weak duality, dual optimal value $\geq c^T x$, and contradicts with unbounded dual.

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**Certificate of Optimality**

• **Corollary.** Given primal feasible sol $x^*$ and dual feasible sol $y^*$, with primal obj = dual obj, then $x^*$ optimal for (P) and $y^*$ optimal for (D).

• **Proof:**
  
  • Have $c^T x^* = b^T y^*$.
  
  • $c^T x \leq b^T y^* = c^T x^*$ by weak duality, and $x^*$ must be optimal for (P).
  
  • Similar argument for $y^*$ in (D).

Can check optimality from primal and dual feasibility, and comparing objective values.
Suppose have a feasible primal LP…

Will there ever be a gap between the optimal dual value and the optimal primal value?

No! Strong Duality Theorem

- **Theorem.** If an LP has an optimal solution then so does its dual, and the two optimal values are equal.

The final tableau in simplex method proves that the BFS is optimal. *(How?)*

-> We use simplex method to prove strong duality.
Review: Tableau from a Basis

- Construct tableau for basis \( B \) from:
  \[
  \begin{align*}
  \max \quad & z \\
  \text{s.t.} \quad & z - c^T x = 0 \\
  & A x = b, \quad x \geq 0 
  \end{align*}
  \]

- Obtain (see lecture 5):
  \[
  z + c_B^T A_B^{-1} A_{B'} x_{B'} - c_{B'}^T x_{B'} = c_B^T A_B^{-1} b \\
  I x_B + A_B^{-1} A_{B'} x_{B'} = A_B^{-1} b
  \]

Using Simplex to Solve Dual

- **Lemma.** Let \( B \) denote optimal basis of primal LP in standard equality form. Then \( y = c_B^T A_B^{-1} \) is optimal dual sol.
  \[
  \begin{align*}
  \max \quad & c^T x \quad \text{(P)} \\
  \text{s.t.} \quad & A x = b \\
  & x \geq 0 \\
  \min \quad & b^T y \quad \text{(D)} \\
  \text{s.t.} \quad & A^T y \geq c
  \end{align*}
  \]

- **Proof.** Let \( x^* \) denote optimal primal solution.
  \[
  z + (c_B^T A_B^{-1} A_{B'} - c_{B'}^T) x_{B'}^* = c_B^T A_B^{-1} b, \quad \text{and so}
  \]
  \[
  z + (y^T A_{B'} - c_{B'}^T) x_{B'}^* = y^T b \quad \text{(*)} \quad \text{(subst}
  \]
  \[
  y^T = c_B^T A_B^{-1} \}
  \]
  \[
  \{\text{mult by } A_B; \text{ take transp.}\}
  \]
  
  Have: \( A_B^T y = c_B \)
  
  \( y^T A_{B'} - c_{B'}^T \geq 0 \) because \( B \) is optimal, and so \( A_{B'}^T y \geq c_{B'} \)
  
  So: \( A^T y \geq c \), and \( y \) is **dual feasible**.

  By (*): \( c^T x^* = z = y^T b = b^T y \). By weak duality, since \( x^* \) is optimal
  
  for (P), then \( y \) is **optimal** for (D).
Strong Duality Theorem

- **Theorem.** If a primal LP has an optimal solution then so does its dual, and the two optimal values are equal.

- **Proof.** (Prove here for standard equality form)
  - Primal LP has an optimal solution, and thus an optimal BFS
  - By “duality lemma” (last slide), optimal primal solution and optimal dual solution have same objective value.

Example: Furniture problem

- max \( z = 60x_1 + 30x_2 + 20x_3 \)
  s.t. \( 8x_1 + 6x_2 + x_3 + x_4 = 48 \)
  \( 4x_1 + 2x_2 + 1.5x_3 + x_5 = 20 \)
  \( 2x_1 + 1.5x_2 + 0.5x_3 + x_6 = 8 \)
  \( x_1, \ldots, x_6 \geq 0 \)
  \( x^* = (2, 0, 8, 24, 0, 0), z = 280 \)

- min \( z = 48y_1 + 20y_2 + 8y_3 \)
  s.t. \( 8y_1 + 4y_2 + 2y_3 \geq 60 \)
  \( 6y_1 + 2y_2 + 1.5y_3 \geq 30 \)
  \( y_1 + 1.5y_2 + 0.5y_3 \geq 20 \)
  \( y_1 \geq 0 \)
  \( y_2 \geq 0 \)
  \( y_3 \geq 0 \)
  \( y^* = (0, 10, 10), z = 280 \)
Example: Furniture problem

- **Optimal (primal) tableau:**

  \[
  \begin{align*}
  z &+ 5x_2 + 10x_5 + 10x_6 = 280 \\
  -2x_2 &+ x_4 + 2x_5 - 8x_6 = 24 \\
  -2x_2 + x_3 + 2x_5 - 4x_6 = 8 \\
  x_1 + 1.25x_2 - 0.5x_5 + 1.5x_6 = 2 \\
  \end{align*}
  \]

  \text{B=\{4,3,1\}}

  \text{x'=\{2,0,8,24,0,0\}}

  \text{z=280}

- **Dual solution:** \( y^T = c_B^TA_B^{-1} \)

  \[
  A_B = \begin{pmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{pmatrix} \quad A_B^{-1} = \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix}
  \]

  \[
  (y_1, y_2, y_3)^T = (0, 20, 60)^T (1, 2, -8) = (0, 10, 10)^T
  \]

  \[
  \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix}
  \]

**Note 1:** \( c_B \) and \( A_B \) are from original standard equality form, using basis \( B \) from final tableau

**Note 2:** The order of columns of \( A_B \) is the same as \( \{4,3,1\} \). We also see that \( A_B^{-1} \) can also be read directly from the final tableau!

Remark: “Reading” the optimal dual

- The final primal tableau also directly gives the optimal dual sol:

  \[
  z + 5x_2 + 10x_5 + 10x_6 = 280
  \]

  \( x_4, x_5, \) and \( x_6 \) are slack vars in original primal tableau, correspond to constraints with dual vars \( y_1, y_2, \) and \( y_3 \) respectively.

- Can read off \( (y_1, y_2, y_3)^T = (0, 10, 10)^T \) directly from the “z equation” in final tableau

- Careful: the original primal tableau might have flipped the order of the first two constraints. In this case, the order of variables in the “z equation” in the final primal tableau would be:

  \[
  z + 5x_2 + 10x_5 + 0x_4 + 10x_6 = 280
  \]

- Now we’d read off the optimal dual solution as \( (y_1, y_2, y_3)^T = (0, 10, 10)^T \) by being careful to use the correct coefficients.
• **Theorem.** The dual of the dual of an LP is the original LP.

• Can establish this knowing nothing more than the rules for duality. Exercise.

Summary: Duality

<table>
<thead>
<tr>
<th>PRIMAL \ DUAL</th>
<th>Infeasible</th>
<th>Has optimal solution</th>
<th>Unbounded</th>
</tr>
</thead>
<tbody>
<tr>
<td>Infeasible</td>
<td>possible</td>
<td>impossible</td>
<td>possible</td>
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</tr>
<tr>
<td>Unbounded</td>
<td>possible</td>
<td>impossible</td>
<td>impossible</td>
</tr>
</tbody>
</table>

(a) consequence of weak duality
(b) consequence of strong duality
(c) e.g., a primal LP with an optimal solution
(d) e.g., an unbounded primal LP
(e) e.g., an unbounded dual LP
(f)
Primal and Dual Infeasible

• max $2x_1 - x_2$
  s.t. $x_1 - x_2 \leq 1$
  $-x_1 + x_2 \leq -2$
  $x_1, \ x_2 \geq 0$

• min $y_1 - 2y_2$
  s.t. $y_1 - y_2 \geq 2$
  $-y_1 + y_2 \geq -1$
  $y_1, \ y_2 \geq 0$

Summary

• Every primal problem has a dual problem, and the dual of the dual is the primal problem

• Optimal dual solutions provide economic intuition and allow sensitivity analysis

• Weak duality theorem: feasible primal value $\leq$ feasible dual value
  – Certificate of optimality
  – Primal unbounded -> dual infeasible

• Strong duality thm: if LP has solution, then optimal primal value = optimal dual value