

AM 121: Intro to Optimization Models and Methods Fall 2018

Lecture 7: LP Duality



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Lesson Plan

- Certificate of optimality
- Primal and Dual LPs
- Weak duality theorem
- Strong duality theorem

Jensen & Bard: 4.2 before Complementary Slackness

Review

- **Standard equality form**

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- **Standard inequality form**

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

LP Duality

- Linear programs come in **primal/dual pairs**
- Why is *LP duality* useful?
 - it leads to the **dual simplex** method, which has practical impact
 - it provides **new insight** into problems
 - it allows for **sensitivity analysis** (*next lecture*)
 - it provides a new way to reason about **optimality and infeasibility**

Establishing Optimality

- $\max x_1 - x_2 + 7x_3$
s.t. $2x_1 - x_2 + x_3 = 1$
 $x_1 + x_2 + 2x_3 = 5$
 $x_1, x_2, x_3 \geq 0$
- Every feasible solution provides a lower bound of the optimal objective value
- Suppose find $x=(0,1,2)$, value 13.
- Optimal val ≥ 13 .
- *But how far away is $x=(0,1,2)$ from optimality?*

Establishing optimality

- $\max x_1 - x_2 + 7x_3$
s.t. $2x_1 - x_2 + x_3 = 1$ (a)
 $x_1 + x_2 + 2x_3 = 5$ (b)
 $x_1, x_2, x_3 \geq 0$
- (P)**

$x=(0,1,2)$, value 13 Optimal?

- Example row operations:
$$\begin{array}{rcl}
 & 2x_1 - x_2 + x_3 = 1 & (a) \\
 + & \underline{3(x_1 + x_2 + 2x_3) = 3(5)} & 3(b) \\
 & 5x_1 + 2x_2 + 7x_3 = 16 & \\
 & x_1 - x_2 + 7x_3 \leq 5x_1 + 2x_2 + 7x_3 = 16 &
 \end{array}$$
 - Consider $3(a) + 2(b)$:
$$\begin{array}{rcl}
 & 3(2x_1 - x_2 + x_3) = 3(1) & 3(a) \\
 + & \underline{2(x_1 + x_2 + 2x_3) = 2(5)} & 2(b) \\
 & 8x_1 - x_2 + 7x_3 = 13 & \\
 & x_1 - x_2 + 7x_3 \leq 8x_1 - x_2 + 7x_3 = 13 &
 \end{array}$$
 - Conclude $x=(0,1,2)$ is **optimal**.

Certificate of Optimality

- Convince a friend that x is optimal by presenting “optimal constraint multipliers” $y^*=(3,2)$.
- Friend can follow these steps:
 - Verify x is feasible
 - Verify objective value $x =$ upper bound implied by multipliers y^*
- y^* provide a **certificate of optimality** for x^*
- **The dual LP finds these optimal multipliers**

- Adding constraints, **any** feasible solution x must satisfy:

$$y_1(2x_1 - x_2 + x_3) + y_2(x_1 + x_2 + 2x_3) = y_1 + 5y_2$$

$$\rightarrow (2y_1+y_2)x_1 + (-y_1 + y_2)x_2 + (y_1+ 2y_2)x_3 = y_1 + 5y_2$$
- For an *upper bound*, we need:

$$x_1 - x_2 + 7x_3 \leq (2y_1+y_2)x_1 + (-y_1+y_2)x_2 + (y_1+2y_2)x_3 = y_1 + 5y_2$$
- Since $x \geq 0$, sufficient $1 \leq (2y_1 + y_2)$; $-1 \leq (-y_1 + y_2)$; $7 \leq (y_1 + 2y_2)$.

- Can solve:

$$\begin{array}{ll} \min & y_1 + 5y_2 & \text{(D)} \\ \text{s.t.} & 2y_1 + y_2 \geq 1 \\ & -y_1 + y_2 \geq -1 \\ & y_1 + 2y_2 \geq 7 \end{array}$$

- This is the **dual** for the original (**primal**) LP!
- By construction: optimal value of **(D)** \geq optimal value of **(P)**.
- In the example: $y^*=(3,2)$ is feasible for **(D)** with value 13; and $x^*=(0,1,2)$ is feasible for **(P)** with value 13.
- Conclude that y^* optimal for **(D)**; x^* optimal for **(P)**.

Example: Primal and Dual LPs

- **max** $x_1 - x_2 + 7x_3$ (P)
 s.t. $2x_1 - x_2 + x_3 = 1$
 $x_1 + x_2 + 2x_3 = 5$
 $x_1, x_2, x_3 \geq 0$
- **min** $y_1 + 5y_2$ (D)
 s.t. $2y_1 + y_2 \geq 1$
 $-y_1 + y_2 \geq -1$
 $y_1 + 2y_2 \geq 7$

- *Going from the Primal to the Dual:*
 - coefficients primal objective \rightarrow coefficients dual RHS
 - coefficients primal RHS \rightarrow coefficients dual objective
 - dual “A matrix” is transpose of primal “A matrix”
 - # primal variables = # dual constraints
 - non-negative primal variables \rightarrow “ \geq ” dual constraints
 - # primal constraints = # dual variables
 - equality primal constraints \rightarrow “free” dual variables

Primal/Dual for Standard Equality Form

- **max** $c^T x$ (P)
 s.t. $Ax = b$
 $x \geq 0$
- **min** $b^T y$ (D)
 s.t. $A^T y \geq c$

Primal	Dual
equality constraints	free variables
non-negative variables	\geq constraints

Example: Furniture problem (1 of 2)

- Make desks, tables and chairs.
- Resources: 48 lumber hours, 20 finishing hours, 8 carpentry hours.
- Desk profit \$60, table \$30 and chair \$20.

	Desk	Table	Chair
Lumber	8	6	1
Finishing	4	2	1.5
Carpentry	2	1.5	0.5

Amount of each resource needed to make each type of furniture

Example: Furniture problem (2 of 2)

- $\max z = 60x_1 + 30x_2 + 20x_3$
s.t. $8x_1 + 6x_2 + x_3 + x_4 = 48$
 $4x_1 + 2x_2 + 1.5x_3 + x_5 = 20$
 $2x_1 + 1.5x_2 + 0.5x_3 + x_6 = 8$
 $x_1, \dots, x_6 \geq 0$

$x^* = (2, 0, 8, 24, 0, 0), z = 280$

- $\min z = 48y_1 + 20y_2 + 8y_3$
s.t. $8y_1 + 4y_2 + 2y_3 \geq 60$
 $6y_1 + 2y_2 + 1.5y_3 \geq 30$
 $y_1 + 1.5y_2 + 0.5y_3 \geq 20$
 $y_1 \geq 0$
 $y_2 \geq 0$
 $y_3 \geq 0$

} Constraints from A^Tb.
Otherwise, y vars would be free.

$y^* = (0, 10, 10), z = 280$

Interpretation of Optimal Dual

- The optimal multiplier y_i is the **shadow price** on the i -th constraint: the amount by which primal objective increases if RHS b_i increases by 1 (while the current basis remains optimal).
- In the furniture example, this represents the value of additional resources:
 - lumber = \$0
 - finishing = \$10
 - carpentry = \$10

Additional supply of lumber not useful (it is already in surplus in optimal sol!)

(Defer details until next lecture)

Constructing the Dual for a General LP

Primal

- $\max c^T x + d^T w$
 s.t. $Ax + Ew = b$ (y_i)
 $Fx + Gw \leq e$ (u_j)
 $x \geq 0, w$ free

Dual

- $\min b^T y + e^T u$
 s.t. $A^T y + F^T u \geq c$ (x_k)
 $E^T y + G^T u = d$ (w_j)
 y free, $u \geq 0$

Observations

$(A \ E \rightarrow \begin{matrix} A^T & F^T \\ F & G \end{matrix})$
 $(E^T \ G^T)$
 obj coeff. \rightarrow RHS coeff.
 equalities \rightarrow free vars
 inequal \rightarrow nonneg vars
 free vars \rightarrow equalities
 nonneg vars \rightarrow inequal

Example: Primal-Dual Pair

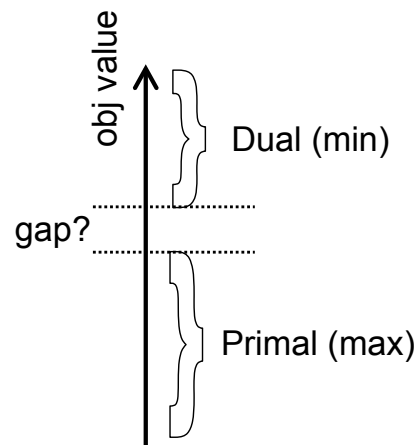
- $\max x_1 - x_2$
s.t. $-3x_1 + x_2 = 2$ y_1 free
 $2x_1 - x_2 \leq -1$ y_2 nonnegative
 $x_1 \geq 0, x_2$ free
- $\min 2y_1 - y_2$
s.t. $-3y_1 + 2y_2 \geq 1$ x_1 nonnegative
 $y_1 - y_2 = -1$ x_2 free
 y_1 free, $y_2 \geq 0$

Primal/Dual for Standard Inequal. Form

- $\max c^T x$ (P)
s.t. $Ax \leq b$
 $x \geq 0$
- $\min b^T y$ (D)
s.t. $A^T y \geq c$
 $y \geq 0$

Obtain this by setting $\{d, A, E, G\}$ to zero in recipe

Weak Duality Theorem



Theorem (weak duality): The objective value of any feasible primal solution is weakly less than the objective value of any feasible dual solution.

Weak Duality Theorem

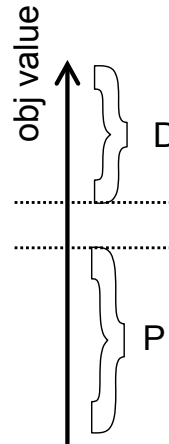
- **Theorem.** The objective value of any feasible primal solution is weakly less than the objective value of any feasible dual solution.
- **Proof.** (prove for standard equality form)

$$\begin{array}{ll}
 \max & c^T x \quad (P) \\
 \text{s.t.} & Ax = b \\
 & x \geq 0
 \end{array}
 \qquad
 \begin{array}{ll}
 \min & b^T y \quad (D) \\
 \text{s.t.} & A^T y \geq c
 \end{array}$$

Consider x feasible for (P) and y feasible for (D).

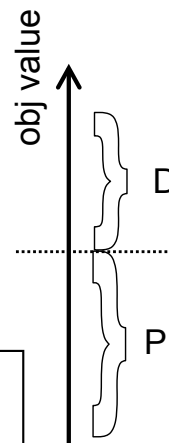
$$c^T x \leq (A^T y)^T x = (y^T A) x = y^T A x = y^T b = b^T y$$

- **Corollary.** If a primal LP is unbounded then its dual problem is infeasible.
- **Proof.** Suppose for contradiction primal is unbounded and (D) has a feasible solution y . By weak duality, primal optimal value $\leq b^T y$. Contradiction.
- **Corollary.** If a dual LP is unbounded then its primal problem is infeasible.
- **Proof.** Suppose for contradiction dual is unbounded and (P) has a feasible solution x . By weak duality, dual optimal value $\geq c^T x$. Contradiction.

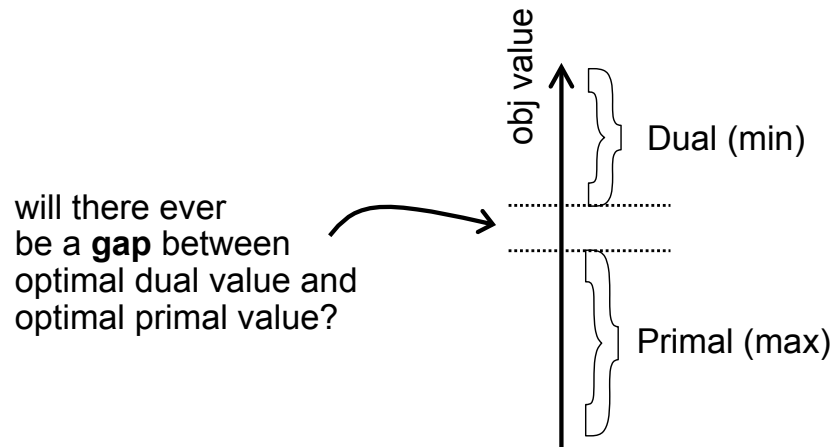


Certificate of Optimality

- **Corollary.** Given primal x^* and dual y^* with primal obj = dual obj, then x^* is optimal for (P) and y^* is optimal for (D).
- **Proof:**
- Have $c^T x^* = b^T y^*$.
- For any x , have $c^T x \leq b^T y^* = c^T x^*$ by weak duality. So, x^* is optimal for (P).
- Similar argument for y^* in (D).

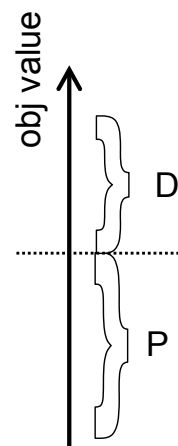


Can check optimality from primal and dual feasibility and comparing objective values.



No! Strong Duality Theorem

- **Theorem (Strong Duality).** If an LP has an optimal solution then so does its dual, and the optimal values are equal.



Review: Initial LP -> Tableau

- Construct tableau for basis B from initial LP:

$$\begin{aligned} \max \quad & z \\ \text{s.t.} \quad & z - c^T x = 0 \\ & Ax = b, x \geq 0 \end{aligned}$$

- Obtain (see Lecture 5):

$$\begin{aligned} z + c_B^T A_B^{-1} A_{B'} x_{B'} - c_{B'}^T x_{B'} &= c_B^T A_B^{-1} b \\ I x_B + A_B^{-1} A_{B'} x_{B'} &= A_B^{-1} b \end{aligned}$$

Duality Lemma

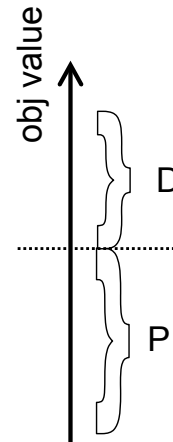
- **Lemma.** If B is the **optimal** basis of a primal LP in standard equality form, then $y^T = c_B^T A_B^{-1}$ is the optimal dual solution.

$$\begin{array}{ll} \max & c^T x \quad (P) \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \min & b^T y \quad (D) \\ \text{s.t.} & A^T y \geq c \end{array}$$

- **Proof.** Let x^* denote optimal primal solution. Let $y^T = c_B^T A_B^{-1}$
 $z + (c_B^T A_B^{-1} A_{B'} - c_{B'}^T) x_{B'}^* = c_B^T A_B^{-1} b$, and so
 $z + (y^T A_{B'} - c_{B'}^T) x_{B'}^* = y^T b \quad (*) \quad \{\text{subst } y^T = c_B^T A_B^{-1}\}$
 Have $y^T A_{B'} - c_{B'}^T \geq 0$ because B is optimal, and so $A_{B'}^T y \geq c_{B'}$.
 Since $A_B^T y = c_B$, have $A^T y \geq c$, and y is **dual feasible**.
 By (*): $c^T x^* = z = y^T b = b^T y$. By weak duality, since x^* is **optimal** for (P), then y is **optimal** for (D).

Strong Duality Theorem

- **Theorem (Strong Duality).** If an LP has an optimal solution then so does its dual, and the optimal values are equal.
- **Proof.** (for standard equality form)
 - Primal LP has an optimal solution, and thus an optimal BFS.
 - By duality lemma, the optimal dual solution has the same objective value as the optimal primal solution.



Example: Furniture problem

- $\max z = 60x_1 + 30x_2 + 20x_3$
 s.t. $8x_1 + 6x_2 + x_3 + x_4 = 48$
 $4x_1 + 2x_2 + 1.5x_3 + x_5 = 20$
 $2x_1 + 1.5x_2 + 0.5x_3 + x_6 = 8$
 $x_1, \dots, x_6 \geq 0$
 $x^* = (2, 0, 8, 24, 0, 0), z = 280$
- $\min z = 48y_1 + 20y_2 + 8y_3$
 s.t. $8y_1 + 4y_2 + 2y_3 \geq 60$
 $6y_1 + 2y_2 + 1.5y_3 \geq 30$
 $y_1 + 1.5y_2 + 0.5y_3 \geq 20$
 $y_1 \geq 0$
 $y_2 \geq 0$
 $y_3 \geq 0$
 $y^* = (0, 10, 10)^T, z = 280$

Example: Furniture problem

- **Optimal (primal) tableau:**

$$\begin{array}{rclcl}
 z & +5x_2 & & +10x_5 + 10x_6 & = 280 \\
 & -2x_2 & + 1x_4 & + 2x_5 - 8x_6 & = 24 \quad B=\{4,3,1\} \\
 & -2x_2 + x_3 & & + 2x_5 - 4x_6 & = 8 \quad x^*=(2,0,8,24,0,0), \\
 x_1 + 1.25x_2 & & & -0.5x_5 + 1.5x_6 & = 2 \quad z=280
 \end{array}$$

- **Dual solution: $y^T = c_B^T A_B^{-1}$**

- $A_B = \begin{pmatrix} 1 & 1 & 8 \\ 0 & 1.5 & 4 \\ 0 & 0.5 & 2 \end{pmatrix}$ $A_B^{-1} = \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix}$

Note 1: The order of columns in defining A_B and c_B is 4, 3, 1 because row 1 in the final primal tableau isolates x_4 , row 2 isolates x_3 and row 3 isolates x_1 .

Note 2: The values of A_B^{-1} can also be read directly from final tableau (from cols. on INITIAL isolated variables). Optimal dual can also be read from final tableau.

- $(y_1, y_2, y_3) = (0, 20, 60)$ $\begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix} = (0, 10, 10)$

Reading Dual from Final Tableau

- The final primal tableau also gives optimal dual sol:

$$z \quad +5x_2 \quad +0x_4 +10x_5 +10x_6 = 280$$

- If x_j is basic in the initial tableau then reduced cost c_j in final tableau gives dual value y_j corresponding to the primal equality

- Read off $(y_1, y_2, y_3)^T = (0, 10, 10)^T$ from the “z equation”

- *Careful* : read the coefficient for corresponding slack variable, ... they may be out of order. For example, if final tableau is:

$$z \quad +5x_2 \quad +10x_5 +0x_4 +10x_6 = 280$$

then we'd still read optimal dual as $(y_1, y_2, y_3)^T = (0, 10, 10)^T$.

Why can dual be read off final tableau?

- Given an optimal (primal) basis B , then:

$$\text{RHS: } \bar{\mathbf{b}} = \mathbf{A}_B^{-1} \mathbf{b}$$

$$\text{Dual solution: } \mathbf{y}^T = \mathbf{c}_B^T \mathbf{A}_B^{-1}$$

$$\text{Objective value: } z = \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{b} = \mathbf{y}^T \mathbf{b}$$

$$\text{Nonbasic obj coeff: } \bar{\mathbf{c}}_{B'}^T = (\mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_{B'} - \mathbf{c}_{B'}^T)$$

$$\text{For nonbasic } j, \bar{\mathbf{c}}_j = \mathbf{c}_B^T \mathbf{A}_B^{-1} \mathbf{A}_j - \mathbf{c}_j = \mathbf{y}^T \mathbf{A}_j - \mathbf{c}_j$$

- Immediate observations:

- $\bar{c}_j = y_j$ for slack variable x_j since $c_j = 0$ and $A_j = e_j$
(i.e., the j^{th} unit vector) ← can read optimal dual directly from optimal primal tableau
- dual variable y_j is shadow price on RHS of constraint j (since $z = \mathbf{y}^T \mathbf{b}$) ← the optimal dual value gives "shadow price"

Why can \mathbf{A}_B^{-1} be read off final tableau?

Let \tilde{B} and \tilde{B}' denote the isolated and non-isolated variables in the initial tableau.

We have

$$\mathbf{A}_{\tilde{B}} x_{\tilde{B}} + \mathbf{A}_{\tilde{B}'} x_{\tilde{B}'} = \mathbf{b}, \quad (1)$$

with $I = \mathbf{A}_{\tilde{B}}$ by arranging columns according to order of \tilde{B} .

To get to the tableau for the optimal basis B , we multiply (1) by \mathbf{A}_B^{-1} , to obtain

$$\bar{\mathbf{A}} x = \mathbf{A}_B^{-1} \mathbf{A}_{\tilde{B}} x_{\tilde{B}} + \mathbf{A}_B^{-1} \mathbf{A}_{\tilde{B}'} x_{\tilde{B}'} = \mathbf{A}_B^{-1} \mathbf{b} = \bar{\mathbf{b}} \quad (2)$$

In particular, since $I = \mathbf{A}_{\tilde{B}}$ this means we can read off \mathbf{A}_B^{-1} from the coefficients $\bar{\mathbf{A}}_{\tilde{B}}$ ($= \mathbf{A}_B^{-1} \mathbf{A}_{\tilde{B}} = \mathbf{A}_B^{-1}$) in the final tableau that correspond to the columns of the initial isolated variables \tilde{B} .

- **Theorem.** The dual of the dual of an LP is the original LP.
- Can establish this knowing nothing more than the rules for duality. Exercise.

Summary: Duality

PRIMAL \ DUAL	infeasible	has optimal solution	unbounded
infeasible	possible	impossible	possible
has optimal solution	impossible	possible	impossible
unbounded	possible	impossible	impossible

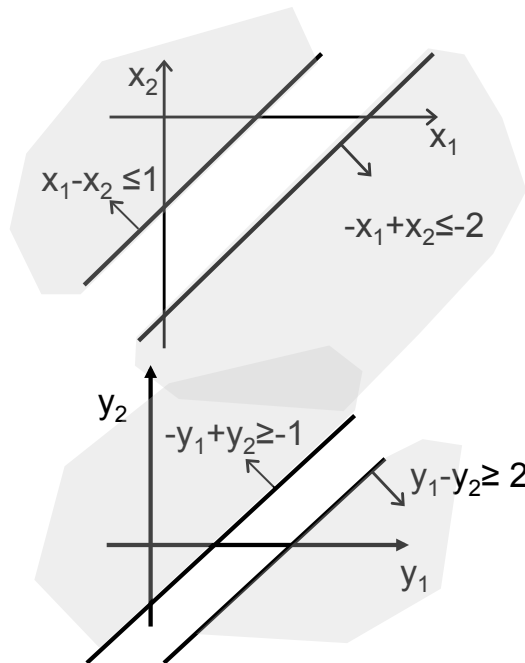
(f) points to the top-left cell (infeasible, possible).
 (b) points to the top-middle cell (has optimal solution, impossible).
 (e) points to the top-right cell (unbounded, possible).
 (d) points to the bottom-left cell (unbounded, possible).
 (c) points to the bottom-middle cell (unbounded, impossible).
 (a) points to the bottom-right cell (unbounded, impossible).

- (a) consequence of weak duality
- (b) consequence of strong duality
- (c) e.g., a primal LP with an optimal solution
- (d) e.g., an unbounded primal LP
- (e) e.g., an unbounded dual LP
- (f)

Primal and Dual Infeasible

- $$\begin{aligned} &\max 2x_1 - x_2 \\ &\text{s.t. } x_1 - x_2 \leq 1 \\ &\quad -x_1 + x_2 \leq -2 \\ &\quad x_1, x_2 \geq 0 \end{aligned}$$

- $$\begin{aligned} &\min y_1 - 2y_2 \\ &\text{s.t. } y_1 - y_2 \geq 2 \\ &\quad -y_1 + y_2 \geq -1 \\ &\quad y_1, y_2 \geq 0 \end{aligned}$$



Summary: LP Duality

- Every primal problem has a dual problem, and the dual of the dual is the primal problem
- Optimal dual solutions provide economic intuition (and allow sensitivity analysis, next lecture).
- **Weak duality theorem:** feasible primal value \leq feasible dual value
 - Certificate of optimality
 - Primal unbounded \Rightarrow dual infeasible
- **Strong duality thm:** if LP has solution, then optimal primal value = optimal dual value