AM 121: Intro to Optimization Models and Methods
Fall 2018

Lecture 6: Phase I, degeneracy, smallest subscript rule.

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Lesson Plan

• Simplex method review and proof of termination
• Phase 1 (initialization)
• Degeneracy, cycling, smallest subscript rule.
• The Fundamental Theorem of Linear Programming.

Textbook Readings: 3.7 and 3.8
Review: A Tableau

\[
\begin{align*}
\text{max} \quad & z \\
\text{s.t.} \quad & \begin{aligned}
  z - c^T x &= 0 \\
  Ax &= b \\
  x &\geq 0
\end{aligned}
\end{align*}
\]

**Definition.** The tableau for basis \( B \) is a system of eqns where the **basic variables are isolated**.

- For basis \( B \) (with \( B' = N \setminus B \)) the tableau is:
  \[
  z + \bar{c}_B^T \bar{x}_{B'} = \bar{v} \\
  I \bar{x}_B + \bar{A}_{B'} \bar{x}_{B'} = \bar{b}
  \]

**Example**

- max \( z = x_1 + x_2 \)
- s.t. \( x_1 \leq 2 \)
  \( x_1 + 2x_2 \leq 4 \)
  \( x_1, \ x_2 \geq 0 \)

- max \( z = x_1 + x_2 \)
- s.t. \( x_1 + x_3 = 2 \)
  \( x_1 + 2x_2 + x_4 = 4 \)
  \( x_1, \ x_2, \ x_3, \ x_4 \geq 0 \)

- Initial tableau (for basis \( \{3,4\} \)):
  \[
  \begin{align*}
  z - x_1 - x_2 &= 0 \\
  x_1 + x_3 &= 2 \\
  x_1 + 2x_2 + x_4 &= 4
  \end{align*}
  \]
Example of Simplex Method

\[ z - x_1 - x_2 = 0 \]
\[ \text{Basic } x_3, x_4 \]

\[ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + x_3 = 2 \]
\[ x_1 + 2x_2 + x_4 = 4 \]
\[ \text{Ratio } 2/1, 4/1 \]
\[ x_1 \text{ to enter. } x_3 \text{ to leave. pivot}(3,1) \]

\[ z - x_2 + x_3 = 2 \]
\[ x_1 + x_3 = 2 \]
\[ \begin{bmatrix} 2x_2 \end{bmatrix} - x_3 + x_4 = 2 \]
\[ \text{Basic } x_1, x_4 \]
\[ \text{Ratio } 2/2 \]
\[ x_2 \text{ to enter. } x_4 \text{ to leave. pivot}(4,2) \]

\[ z + \frac{1}{2} x_3 + \frac{1}{2} x_4 = 3 \]
\[ x_1 + x_3 = 2 \]
\[ x_2 - \frac{1}{2} x_3 + \frac{1}{2} x_4 = 1 \]
\[ \text{Basic } x_1, x_2 \]
\[ \text{Reduced costs all nonnegative. Stop!} \]

• Solution: \((x_1, x_2, x_3, x_4) = (2, 1, 0, 0), z=3.\)

Comments

• 1. We need to be able to find an initial tableau corresponding to a BFS
• 2. \(\bar{c}_k\) is the reduced cost of nonbasic variable \(x_k\). Amount by which \(z\) decreases when \(\bar{x}_k\) increases (and so \(c_k < 0\) is good).
3. Unboundedness

- $x_i + \sum_{j \in B} \tilde{a}_{ij} x_j = \tilde{b}_i$ (for all $i \in B$)
- Because other nonbasic vars = 0, we can increase $x_k$ while:
  
  $x_i = \tilde{b}_i - \tilde{a}_{ik} x_k \geq 0$ (for all $i \in B$)

- If $\tilde{a}_{ik} \leq 0$ for every $i$ in $B$, then $x_k$ can increase without bound (without affecting objective)!

4. Pivoting to the new Tableau

- **Definition.** A pivot on $(r, k)$ is row operations to construct tableau for $B := B \cup \{k\} \setminus \{r\}$.
- (a) Divide row $x_r + \sum_{j \in B} \tilde{a}_{rj} x_j = \tilde{b}_r$ through by $\tilde{a}_{rk}$ so that coefficient of new basic variable $x_k$ becomes 1.
  
  (Why does RHS of row $r$ remain nonnegative?)

  A: the coefficient $\tilde{a}_{rk}$ is strictly positive!

- (b) Add/subtract multiples of this adjusted row to all other equations (including objective) to remove $x_k$
  
  (Why do these operations not affect isolation of other basic vars?)

  A: the only basic variable with non-zero coefficient in row $r$ is $x_r$.

  (Why does the RHS of the other rows remain nonnegative?)

  A: for a row $r'$ with positive coefficient $\tilde{a}_{r'k}$ we subtract multiple $\frac{\tilde{a}_{r'k}}{\tilde{a}_{rk}} \tilde{b}_r$ of row $r$, and $\left(\frac{\tilde{a}_{r'k}}{\tilde{a}_{rk}}\right) \tilde{b}_r \leq \tilde{b}_{r'}$ by the ratio test.

[Note: we're doing "Gauss-Jordan elimination."]
Degeneracy

- A BFS is **degenerate** if a basic variable $x_i$ has value zero.

- *Ratio test.* $t^* = \min \{ \bar{b}_i / a_{ik} : i \in B, \bar{a}_{ik} > 0 \}$. Pick leaving index $r \in B$ with min ratio.

- If $\bar{a}_{ik} > 0$ and $\bar{b}_i = 0$, then simplex method cannot make the entering variable $x_k$ increase in value.

- Move to an adjacent basis, but without improving objective.

- Ignore this possibility for a moment.

Simplex Termination

- **Theorem.** Simplex method terminates with an optimal solution, or a proof of unboundedness, as long as never reaches a degenerate BFS.

- **Proof.** Suppose LP is **not** unbounded.
  - In every iteration the value of the entering variable $x_k = t^* > 0$, and objective **strictly** increases.
    - $\Rightarrow$ cannot visit same BFS twice.
    - $\Rightarrow$ terminates, since finite number of BFS.

  - If **unbounded**: must reach a tableau that is adjacent to one in which can increase objective without bound.
Remaining Issues

- How to find a first BFS to initialize the simplex method?
- How can we be sure the simplex method will terminate even if there may be degenerate BFSs?

Finding an initial BFS

**Easy case**: If our initial LP in standard inequal. form
\[
\begin{align*}
\text{max} \quad & c^T x \\
\text{s.t.} \quad & Ax \leq b \\
& x \geq 0
\end{align*}
\]
and \( b \geq 0 \), can transform into canonical form by introducing slack variables.

**Example:**
\[
\begin{align*}
\text{max} \quad & x_1 + x_2 \\
\text{s.t.} \quad & x_1 \leq 2 \quad \Rightarrow \quad z - x_1 - x_2 = 0 \\
& x_1 + 2x_2 \leq 4 \quad \Rightarrow \quad x_1 + 2x_2 + x_3 = 2 \\
& x_1, \quad x_2 \geq 0 \\
& x_1 + x_3 + x_4 = 4
\end{align*}
\]
Initialization: General case

- **LP with +ve RHS**, but may have $\geq$ and $=$ constraints

  \[
  \begin{align*}
  \text{max} & \quad 2x_1 + x_2 \\
  \text{s.t.} & \quad x_1 + x_2 \leq 3 \\
  & \quad -x_1 + x_2 \geq 1 \\
  & \quad x_1, x_2 \geq 0
  \end{align*}
  \]

  \[
  \begin{align*}
  \text{max} & \quad 2x_1 + x_2 \\
  \text{s.t.} & \quad x_1 + x_2 + x_3 = 3 \\
  & \quad -x_1 + x_2 \quad -x_4 = 1 \\
  & \quad x_1, x_2, x_3, x_4, x_5 \geq 0
  \end{align*}
  \]

- Don’t have a basis. Not even sure if feasible!

- Introduce “**artificial variable**” $x_5 \geq 0$.

  \[-x_1 + x_2 - x_4 + x_5 = 1\]

- **Auxiliary LP:**

  \[
  \begin{align*}
  \text{min} & \quad x_5 \\
  \text{s.t.} & \quad x_1 + x_2 + x_3 = 3 \\
  & \quad -x_1 + x_2 - x_4 + x_5 = 1 \\
  & \quad x_1, x_2, x_3, x_4, x_5 \geq 0
  \end{align*}
  \]

- **Lemma.** (1) feasible iff (2) has optimal soln with $x_5=0$
  
  \[\implies\] can set $x_5=0$ in (2)
  
  \[\implies\] if opt soln with $x_5=0$, then $x_1\ldots x_4$ feasible for (1)

---

Phase 1 of the simplex method

- Introduce **artificial variables** in “$\geq$” and “$=$” rows. Solve auxiliary problem to check feasibility

- \[
  \begin{align*}
  \text{max} & \quad w \\
  \text{s.t.} & \quad w + x_5 = 0 \quad \text{(a)} \\
  & \quad x_1 + x_2 + x_3 = 3 \quad \text{(b)} \\
  & \quad -x_1 + x_2 - x_4 + x_5 = 1 \quad \text{(c)} \\
  & \quad x_1, \ldots, x_5 \geq 0
  \end{align*}
  \]

- **Why did this help?** Easy BFS for auxiliary LP!

- $x_3$ but not $x_5$ isolated. To isolate $x_5$ can use (a) - (c).

- Get tableau for $B=\{3,5\}$:

  \[
  \begin{align*}
  w + x_1 - x_2 + x_4 &= -1 \\
  x_1 + x_2 + x_3 &= 3 \\
  -x_1 + x_2 - x_4 + x_5 &= 1
  \end{align*}
  \]

- Can now solve with simplex. If obtain $w=0$, **can find an initial BFS for original problem.**
Phase 1-Phase 2 Example (1 of 2)

\[ \begin{align*}
w + x_1 &- x_2 + x_4 = -1 \quad (a) & \quad \text{Basic } x_3, x_5 \\
x_1 + x_2 + x_3 & = 3 \quad (b) & \quad \text{Ratio } 3/1 \ 1/1 \\
\downarrow & -x_1 + \boxed{x_2} - x_4 + x_5 = 1 \quad (c) & \quad x_3 \text{ to enter, } x_5 \text{ to leave}
\end{align*} \]

\[ \begin{align*}
w + x_5 & = 0 \quad (*) & \quad B=\{2,3\}. \text{ Optimal.} \\
2x_1 + x_3 + x_4 & = 2 \quad (b) \\
-x_1 + x_2 - x_4 & = 1
\end{align*} \]

- Can we find a BFS for original LP?
- Drop (*) and \(x_5\) (since \(x_5=0\)), and obtain system:
  \[ \begin{align*}
2x_1 + x_3 + x_4 & = 2 \\
-x_1 + x_2 - x_4 & = 1
\end{align*} \]

As long as final BFS is non-degenerate, \(x_5 (=0)\) will be non-basic and we have a basis for the original LP (\(\{2,3\}\)).

Phase 1-Phase 2 Example (2 of 2)

\[ \begin{align*}
z - 2x_1 - x_2 & = 0 \quad (a) & \quad \text{original obj} \\
2x_1 + x_3 + x_4 & = 2 \quad (b) \\
-x_1 + x_2 - x_4 & = 1 \quad (c)
\end{align*} \]

- Need to isolate \(\{x_2,x_3\}\). Do (a) + (c). \textbf{Now begin Phase 2.}

\[ \begin{align*}
z - 3x_4 \quad - x_4 & = 1 \quad \text{Basic } x_2, x_3 \\
\downarrow & 2x_1 + x_3 + x_4 \quad = 2 \quad \text{Ratio } 2/2 \\
\uparrow & -x_1 + x_2 - x_4 \quad = 1
\end{align*} \]

\[ \begin{align*}
z + \frac{1}{2} x_3 + \frac{1}{2} x_4 & = 4 \\
x_1 + \frac{1}{2} x_3 + \frac{1}{2} x_4 & = 1 \\
x_2 + \frac{1}{2} x_3 - \frac{1}{2} x_4 & = 2
\end{align*} \]

\[ z=4 \]
Summary: Phase 1

• Introduce **artificial variables** in “≥” and “=” rows

• Solve **auxiliary LP** to find solution with all artificial variables taking on value zero

• If exists, then this solution **provides a BFS for the original LP**. Else, original LP is infeasible.

• A key property of the auxiliary LP is that it has a BFS that is easy to identify.

Degeneracy

• **Definition.** A **basic solution** is **degenerate** when one or more basic variables have value zero.

• **Definition.** A **tableau** is **degenerate** when one or more RHS values \( b_i \) have value zero

• Whenever we have to choose between several leaving indices, the next tableau is degenerate…
Example (Degeneracy)

\[ \text{max } 2x_1 + x_2 \]
\[ \text{s.t. } x_1 - x_2 \leq 1 \]
\[ x_1 \leq 1 \]
\[ x_2 \leq 1 \]
\[ x_1, x_2 \geq 0 \]

Initial tableau (non degenerate):

\[ z - 2x_1 - x_2 = 0 \]
\[ x_1 - x_2 + x_3 = 1 \]
\[ x_1 + x_4 = 1 \]
\[ x_2 + x_5 = 1 \]

\[ B = \{3, 4, 5\} \]
\[ x = (0, 0, 1, 1, 1) \]

Example (Degeneracy)

\[ z - 2x_1 - x_2 = 0 \]
\[ x_1 - x_2 + x_3 = 1 \]
\[ x_1 + x_4 = 1 \]
\[ x_2 + x_5 = 1 \]

Basic \( x_3, x_4, x_5 \)
Ratio \( \frac{1}{1} \quad \frac{1}{1} \)
\( x_1 \) to enter, \( x_3 \) to leave (tie break)

\[ x = (1, 0, 0, 0, 1) \]
\( x_1 = 0 \), degenerate!
\( x_3 \) to enter, \( x_4 \) to leave

\[ z - 3x_2 + 2x_3 = 2 \]
\[ -x_2 + x_3 = 1 \]
\[ x_2 - x_3 + x_4 = 0 \]
\[ x_2 + x_5 = 1 \]

Basic \( x_1, x_4, x_5 \)
Ratio \( \frac{0}{1} \quad \frac{1}{1} \)
\( x = (1, 0, 0, 0, 1) \)
\( x_1 = 0 \), degenerate!
\( x_2 \) to enter, \( x_4 \) to leave

\[ z - x_3 + 3x_4 = 2 \]
\[ x_1 - x_3 + x_4 = 1 \]
\[ x_2 - x_3 + x_4 = 0 \]
\[ -x_4 + x_5 = 1 \]

Basic \( x_1, x_2, x_3 \)
Ratio \( \frac{1}{1} \)
\( x = (1, 0, 0, 1) \)
\( x_1 = 0 \), degenerate!
\( x_3 \) to enter, \( x_5 \) to leave

Basis and tableau has changed, but BFS and obj value unchanged

\[ z + 2x_4 + x_5 = 3 \]
\[ x_1 + x_4 = 1 \]
\[ x_2 + x_5 = 1 \]
\[ x_3 - x_4 + x_5 = 1 \]

Optimal solution! Was OK here
• 5 vars, 3 equations. Each basic solution adds n-m=2 additional binding constraints (nonbasic vars = 0), implies unique solution.

• Degeneracy occurs when more than (n-m) constraints intersect at an extreme point (e.g., point (1,0).)

### Degeneracy and Cycling

- Will simplex method terminate?
- Objective value does not strictly increase at each iteration. Earlier proof fails.

- **Definition.** The simplex method **cycles** when it returns to the same tableau
  - E.g., $T_0 \rightarrow T_1 \rightarrow \ldots \rightarrow T_{p-1} \rightarrow T_0$

- In this case, simplex method would **cycle** forever!
From “method” to algorithm:

• Need to make precise remaining design choices

• Choice of entering index:
  – **most negative reduced cost**: choose \( k \in B' \) with smallest \( \bar{c}_k \)
  – **smallest subscript**: choose smallest index \( k \in B' \) with \( \bar{c}_k < 0 \)
  – **random**: choose any \( k \in B' \) with \( \bar{c}_k < 0 \)

• Choice of leaving index: (may be a tie)
  – **smallest subscript**: choose smallest index \( r \in R \)
  – **random**: choose any \( r \in R \)

A Bad Rule

1. Pick the entering variable with the most negative reduced cost (break ties according to index)
2. Pick the exiting variable with the smallest index
Cycling example

- max \( 10x_1 - 57x_2 - 9x_3 - 24x_4 \)  
  s.t. \( 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 \leq 0 \)  
  \( 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 \leq 0 \)  
  \( x_1 \leq 1 \)  
  \( x_1, x_2, x_3, x_4 \geq 0 \)

- Put in canonical form:

\[
\begin{align*}
0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_5 &= 0 \\
0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + x_6 &= 0
\end{align*}
\]

\[
x_1 + x_7 = 1
\]

\[
z = 10x_1 + 57x_2 + 9x_3 + 24x_4 = 0 \quad x_5, x_6, x_7
\]

\[
0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_5 = 0 \quad 0/0.5 \quad 0/0.5 \quad 1/1
\]

\[
x_1
\]

\[
z = 53x_2 - 41x_3 + 204x_4 + 20x_5 = 0 \quad x_1, x_6, x_7
\]

\[
x_1 - 11x_2 - 5x_3 + 18x_4 + 2x_5 = 0 \quad 0/4 \quad 1/11
\]

\[
0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + x_6 = 0
\]

\[
x_1 + x_7 = 1
\]

\[
z = 53x_2 - 41x_3 + 204x_4 + 20x_5 = 0 \quad x_1, x_6, x_7
\]

\[
x_1 - 11x_2 - 5x_3 + 18x_4 + 2x_5 = 0 \quad 0/4 \quad 1/11
\]

\[
0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + x_6 = 0
\]

\[
x_1 + x_7 = 1
\]

\[
z = 10x_1 + 57x_2 + 9x_3 + 24x_4 = 0 \quad x_5, x_6, x_7
\]

\[
0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_5 = 0 \quad 0/0.5 \quad 0/0.5
\]

\[
x_1
\]

\[
z = 10x_1 + 57x_2 + 9x_3 + 24x_4 = 0 \quad x_5, x_6, x_7
\]

\[
0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_5 = 0 \quad 0/0.5 \quad 0/0.5
\]

\[
x_1
\]

\[
z = 10x_1 + 57x_2 + 9x_3 + 24x_4 = 0 \quad x_5, x_6, x_7
\]

\[
0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_5 = 0 \quad 0/0.5 \quad 0/0.5
\]

\[
x_1
\]

where we started!!
Smallest subscript rule

• Entering: amongst those with strictly negative reduced cost, pick var with smallest index.
• Exiting: amongst those with min ratio, pick variable with smallest index.

**Bland’s Theorem.** If the simplex method uses the smallest subscript rule then it will terminate.
• Proof: See Chvatal “Linear Programming” 1983
Fundamental Theorem of LP

• **Theorem.** Any LP has either an optimal solution, is infeasible, or is unbounded.

• **Proof (sketch):**
  – Case 1: Infeasible. (Ok!)
  – Case 2: Unbounded. (Ok!).

• **Note:** some optimization problems do not have this property, e.g. min 1/x s.t. x≥1

Simplex lemma

• Consider an LP in standard equality form (max cᵀx s.t. Ax=b, x≥0) with columns of A that span.

• **Lemma.** If an LP in standard equality form is feasible and bounded, then it has an optimal solution.

• **Proof.** (sketch)
  – If feasible then LP has a BFS (use Phase 1 with smallest subscript rule; must terminate with optimal value zero.)
  – Obtain BFS for original LP from final tableau of Phase 1 (need columns of A to span for this)
  – Simplex with smallest subscript rule for Phase 2. Must terminate. Since not unbounded, must terminate with optimal solution.
Handling Degeneracy in Phase 1?

(Advanced topic)

- Phase 1 must terminate with non-degenerate basic solution to be able to construct BFS for original LP
- Phase 1 may terminate with artificial variable $u_i=0$, but basic. Suppose equation is $\sum_{j=1}^{n} \bar{a}_{ij}x_j + u_i = 0$
- If $\bar{a}_{ij}=0$ for all $j$ then can delete entire equation (redundant constraint)
- Else, some $\bar{a}_{ij}\neq 0$. Pivot on entry $(i,j)$, cause $x_j$ to become basic and variable $u_i$ to become nonbasic.
- Repeat this process until all artificial variables are “driven out” of the (phase 1) basis.

Comments on Optimality

- Consider a BFS $x$.
- If the reduced costs are non-negative, then $x$ is optimal. This is true whether or not $x$ is degenerate. Thus, it is a sufficient test.
- If $x$ is optimal and non-degenerate then the reduced costs will be non-negative. But, a degenerate BFS $x$ can be optimal with negative reduced costs!
- There is no simple test for determining whether a degenerate BFS is optimal.
- The simple test of non-negative reduced costs is sufficient for the simplex method: Bland’s theorem tells us that this optimality test ensures termination, even in the presence of degeneracy.
Comments on Unique Optimality

- Consider a BFS $x$.
- If the reduced costs are positive, then $x$ is the unique optimal solution. This is true whether or not $x$ is degenerate. It is a sufficient test.
- If BFS $x$ is unique optimal and non-degenerate then the reduced costs will be positive.
- But, a degenerate BFS $x$ can be the unique optimal solution but have non-positive reduced costs.
- There is no simple test for determining whether a degenerate BFS is unique optimal.

Summary: Simplex method

- **Phase 1 (auxiliary LP)** can be formulated to find an initial BFS
- Degeneracy (basic variables taking on value zero) occurs when more than $n-m$ constraints intersect on a feasible point
- Cycling can be prevented through the **smallest subscript rule**.
- **Fundamental thm. of LP**: Every LP has an opt. solution, is infeasible, or is unbounded.