AM 121: Intro to Optimization Models and Methods

Lecture 5: Tableau, first look at the simplex algorithm

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Lesson Plan

This lecture: Moving towards an algorithm

• Tableau. Adjacent BFS.
• The simplex method

• A first theorem about the simplex method

Jensen & Bard: 3.4, 3.5, 3.6, 3.10 (for matrix representation)
Warm-up

Canonical form

\[
\begin{array}{cccc}
\text{max } & c^T x \\
\text{s.t.} & Ax=b \\
x \geq 0
\end{array}
\]

Equivalent canonical form

\[
\begin{array}{cccc}
\text{max } & z \\
\text{s.t.} & z - c^T x = 0 \\
& Ax=b \\
x \geq 0
\end{array}
\]

• We’re interested in constructing a **tableau** that corresponds to a basis \( B \)
• Tableau is a system of equations

Example

• \[
\begin{align*}
\text{max } & \quad x_1 + x_2 \\
\text{s.t.} & \quad x_1 + x_3 = 2 \\
& \quad x_1 + 2x_2 + x_4 = 4 \\
& \quad x_1, \ x_2, \ x_3, \ x_4 \geq 0
\end{align*}
\]
• \[
\begin{align*}
\text{max } & \quad z \\
\text{s.t.} & \quad z - x_1 - x_2 = 0 \\
& \quad x_1 + x_3 = 2 \\
& \quad x_1 + 2x_2 + x_4 = 4 \\
& \quad x_1, \ x_2, \ x_3, \ x_4 \geq 0
\end{align*}
\]

• The **tableau** corresponding to basis \( \{3,4\} \) is:
\[
\begin{align*}
z - x_1 - x_2 & = 0 \\
x_1 + x_3 & = 2 \\
x_1 + 2x_2 + x_4 & = 4
\end{align*}
\]
Tableau

\[
\begin{align*}
\text{max} & \quad z \\
\text{s.t.} & \quad z - c^T x = 0 \\
& \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

**Definition.** The **tableau** corresponding to basis \( B \) is a system of eqns where **basic variables are isolated**.

**For basis \( B \) (with \( B' = N / B \)), the tableau is**

\[
\begin{align*}
z + \bar{c}^T x_B &= \bar{v} \\
Ix_B + \bar{A} x_{B'} &= \bar{b}
\end{align*}
\]

- **Basic solution** \( x_B = \bar{b}, x_{B'} = 0 \)
- **Objective** \( z = \bar{v} \)
- **Feasible when** \( \bar{b} \geq 0 \).
- **Parameters** \( \bar{A}, \bar{c}, \bar{v}, \bar{b} \) different from those without bar (RHS can be negative!)

**Constructing a Tableau**

- **Given a basis** \( B \), find tableau:

\[
\begin{align*}
\text{max} & \quad z \\
\text{s.t.} & \quad z - c^T x = 0 \\
& \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
z + \bar{c}^T x_B &= \bar{v} \\
Ix_B + \bar{A} x_{B'} &= \bar{b}
\end{align*}
\]

\[a. \quad \text{Row operations to transform second row}
\]

\[
\begin{align*}
A_B x_B + A_{B'} x_{B'} &= b \\
\Rightarrow I x_B + A_B^{-1} A_{B'} x_{B'} &= A_B^{-1} b \quad (*)
\end{align*}
\]

Let \( \bar{A}'_B = A_B^{-1} A_{B'} \) and \( \bar{b} = A_B^{-1} b \)
Constructing a Tableau

• Given a basis $B$, find tableau:

$$\begin{array}{l}
\text{max } z \\
\text{s.t. } z - c^T x = 0 \\
Ax = b \\
x \geq 0
\end{array}$$

b. Row operations to transform first row

$$z - c_B^T x_B - c_{B'}^T x_{B'} = 0$$

$$c_B^T (x_B + A_B^{-1} A_{B'} x_{B'}) = c_B^T A_B^{-1} b$$

Use (*)

Example: Tableau (1/3)

• $\begin{aligned}
z - x_1 - x_2 & = 0 \\
x_1 + x_3 & = 2 \\
x_1 + 2x_2 + x_4 & = 4
\end{aligned}$

(a) $x=(0,0,2,4)$  
B=$\{3,4\}$

(b) $x=(2,0,0,2)$  
B=$\{1,4\}$

• How obtain tableau for $B=\{1,4\}$?

• Subtract (b) from (c); add (b) to (a):

• $\begin{aligned}
z - x_2 + x_3 & = 2 \\
x_1 + x_3 & = 2 \\
2x_2 - x_3 + x_4 & = 2
\end{aligned}$

• BFS? Objective?

• $x=(2,0,0,2)$ ; $z=2$. An improvement!
Example: Tableau (2/3)

• $z - x_2 + x_3 = 2$
  $x_1 + x_3 = 2$
  $2x_2 - x_3 + x_4 = 2$

How can we improve further?

• Increase $x_2 := t$; keep $x_3 = 0$; $z = 2 + t$
Example: Tableau (2/3)

- $z - x_2 + x_3 = 2$
  - $x_1 + x_3 = 2$
  - $2x_2 - x_3 + x_4 = 2$
  
  x=(2,0,0,2)  
  B={1,4}

How can we improve further?

- Increase $x_2 := t$; keep $x_3=0$; $z = 2+t$
- Feasibility: $x_1=2$ ; $x_4=2 - 2t$.
- Set $x_2 := 1$, because $x_4 < 0$ for $t > 1$.
- New basis $B={1,2}$. $x=(2,1,0,0)$, $z=3$.

Example: Tableau (3/3)

- $z - x_2 + x_3 = 2$ (a)  
  - $x_1 + x_3 = 2$ (b)
  - $2x_2 - x_3 + x_4 = 2$ (c)

  x=(2,0,0,2)  
  B={1,4}

Get tableau for $B={1,2}$

- $z + \frac{1}{2}x_3 + \frac{1}{2}x_4 = 3$  
  - $x_1 + x_3 = 2$
  - $x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_4 = 1$

x=(2,1,0,0)  
B={1,2}

- Obj coeff of nonbasic vars >= 0. Solution optimal!
- Must have $z = 3 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \leq 3$ for any feasible solution, since $x_3, x_4 \geq 0$. 
From Basis to Tableau and Back

• **Theorem.** If set B is a basis for matrix A then there is a tableau corresponding to B. If there is a tableau for set B then B is a basis for A.

• **Proof**
  \( \implies \) perform row operations to isolate the variables \( x_B \) (possible since \( A_B \) is invertible)
  \( \iff \) easy, B is a basis because \( A_B=I \) and therefore spans and is invertible.

**Adjacent Bases**

• **Definition.** Two bases for matrix A are adjacent if they share all but one column

• **Example**
  - \( \max z = x_1 + x_2 \)
  - s.t. \( x_1 + x_3 = 2 \)
  - \( x_1 + 2x_2 + x_4 = 4 \)
  - \( x_1, x_2, x_3, x_4 \geq 0 \)

• Five bases \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{3,4\}.
• B={1,2} is adjacent to {1,3}, not {3,4}. 
The Simplex Method

Work in tableau form
• Step (i). Find an initial BFS.
• Step (ii).
  – Determine if current BFS is optimal.
  – If not, find an adjacent BFS with a larger z-value.
• Step (iii). Return to step (ii), using new BFS as the current BFS.

How fast is this?
• Number of possible BFS is “n choose m”
  \[ \binom{n}{m} = \frac{n!}{(n-m)! \cdot m!} \]
• \( \binom{20}{10} = 184,756 \)
• Even if we can avoid cycling… gets large.
  – “Degeneracy” leads to concern about cycling
  – Next lecture
Empirical performance

• Applied to LPs in canonical form, tends to find an optimal solution after searching less than 3m BFS.
• Example: 20-variable, 10-constraint problem: search ≈ 30 BFS.
• Simplex has very good performance in practice

The Simplex Method

• Work with: max z
  s.t. z - c^T x = 0
  Ax=b, x≥0
• Step 0. Initialization
• Step 1. Check optimality.
• Step 2. Choose entering index.
• Step 3. Check unboundedness.
• Step 4. Choose leaving index.
• Step 5. Pivot to a new tableau. To step 1.
Simplex Method - Overview

- **max** \( z \)
- **s.t.** \( z + \sum_{j \in B'} c_j x_j = \bar{v} \)
  \( x_i + \sum_{j \in B'} \bar{a}_{ij} x_j = \bar{b}_i \) \( (i \in B) \)

- **Step 0.** Initialization (find a BFS and tableau)
- **Step 1.** If \( \bar{c}_j \geq 0 \) for all \( j \in B' \), stop (optimal)
- **Step 2.** Pick entering index \( k \in B' \) with \( \bar{c}_k < 0 \).
- **Step 3.** If \( \bar{a}_{ik} \leq 0 \) for all \( i \in B \), stop (unbounded)
- **Step 4.** Ratio test. \( t^* = \min \{ \frac{\bar{b}_i}{\bar{a}_{ik}} : i \in B, \bar{a}_{ik} > 0 \} \). Pick leaving index \( r \in B \) with min ratio.
- **Step 5.** Pivot on \((r,k)\) entry to get a tableau for \( B := B \cup \{k\} \setminus \{r\} \).

**Example of Simplex**

\[
\begin{align*}
z - x_1 - x_2 &= 0 & \text{Basic } x_3, x_4 \\
\left[ \begin{array}{c}
 x_1 \\
 x_1 + 2x_2 \\
 x_1 + x_3 \\
 2x_2 - x_3 + x_4
\end{array} \right] + x_3 &= 2 & \text{Ratio } 2/1, 4/1 \\
&+ x_4 &= 4 & x_1 \text{ to enter. } x_3 \text{ to leave}
\end{align*}
\]

\[
\begin{align*}
\frac{z}{x} - x_2 + x_3 &= 2 & \text{Basic } x_1, x_4 \\
x_1 + x_3 &= 2 & \text{Ratio } 2/2 \\
\left[ \begin{array}{c}
 2x_2 \\
 x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_4
\end{array} \right] + x_3 &= 2 & x_2 \text{ to enter. } x_4 \text{ to leave}
\end{align*}
\]

\[
\begin{align*}
\frac{z}{x} &+ \frac{1}{2}x_3 + \frac{1}{2}x_4 = 3 & \text{Basic } x_1, x_2 \\
x_1 + x_3 &= 2 & \text{Reduced costs all nonnegative.} \\
x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_4 &= 1 & \text{Stop}
\end{align*}
\]

- Solution: \((x_1, x_2, x_3, x_4) = (2, 1, 0, 0)\), \(z = 3\).
Step 0: Initialization

- Find a basis corresponding to a BFS
  - Details next lecture
- Construct the tableau for this basis

Step 1: Check optimality

- If $\bar{c}_j \geq 0$ (for all j in $B'$) then stop, current solution is optimal.
- Value $z = \bar{v}$. Any feasible solution satisfies
  $$z = \bar{v} - \bar{c}^T x_{B'} \leq \bar{v},$$
  since $\bar{c} \geq 0$ and $x \geq 0$. 
Step 2: Choose Entering Index

- Pick some \( k \) in \( B' \) with \( \bar{c}_k < 0 \) to enter the basis, so that the obj value increases with \( x_k \)

- Say that \( \bar{c}_k \) is the "reduced cost" of nonbasic variable \( x_k \). Amount by which \( z \) decreases when \( x_k \) increases (and so \( \bar{c}_k < 0 \) is good).

Step 3: Check Unboundedness

- \( x_i + \sum_{j \in B'} \bar{a}_{ij} x_j = \bar{b}_i \) (for all \( i \in B \))
- Because other nonbasic vars = 0, we can increase \( x_k \) while:
  \[ x_i = b_i - a_{ik} x_k \geq 0 \] (for all \( i \in B \))
- If \( \bar{a}_{ik} \leq 0 \) for every \( i \) in \( B \), then \( x_k \) can increase without bound! **Stop: objective value is unbounded.**
Step 4: Choose Leaving Index

- Need: \( x_i = \bar{b}_i - \bar{a}_{ik} x_k \geq 0; \) and so \( x_k \leq \frac{\bar{b}_i}{\bar{a}_{ik}} \) (where \( \bar{a}_{ik} > 0 \)).

- **Ratio test:** \( t^* = \min \{ \frac{\bar{b}_i}{\bar{a}_{ik}} : i \text{ in } B, \bar{a}_{ik} > 0 \} \)
- Let R denote basic vars with min ratio
- Pick some \( r \) in R as leaving index; set \( x_k := t^* \)

- New basic variable \( x_k \) “kicks out” the “removed” variable \( x_r \)

Step 5: Pivot to new tableau

- **Definition.** A pivot on variable \( x_k \) in row corresponding to \( x_r \) is row operations to construct tableau for \( B := B \cup \{k\}\setminus\{r\} \).

- (a) Divide row \( x_r + \sum_{j \in B} \bar{a}_{ij} x_j = \bar{b}_r \) through by \( \bar{a}_{rk} \) so that coefficient of new basic variable \( x_k \) becomes 1.  
  *(Why does RHS remain nonnegative?)*

- (b) Add/subtract multiples of this adjusted row to all other equations (including objective) to remove \( x_k \)  
  *(Why do these operations not affect coeffs for other basic vars?)*
From “method” to algorithm:

• Need to make precise remaining design choices

• Choice of entering index:
  – **most negative reduced cost**: choose $k \in B'$ with smallest $\bar{c}_k$
  – **smallest subscript**: choose smallest index $k \in B'$ with $\bar{c}_k < 0$
  – **random**: choose any $k \in B'$ with $\bar{c}_k < 0$

• Choice of leaving index: (may be a tie)
  – **smallest subscript**: choose smallest index $r \in R$
  – **random**: choose any $r \in R$

• These details matter; e.g., to prevent cycles
  – next lecture

Degeneracy

• For now we assume this doesn’t happen, but (looking ahead), we say a BFS is “**degenerate**” if one or more basic variables have value zero.

• In this case, $t^* = 0$ and simplex cannot make the entering variable increase in value.

• Rather, simplex considers an adjacent basis without improving the objective.
Simplex Termination

- **Theorem.** If never reach a degenerate BFS, then the simplex method terminates with an *optimal* solution, or a *proof of unboundedness*.

- **Proof.** Suppose LP is *not* unbounded.
  - In every iteration, by non-degeneracy, the value of the entering variable $x_k := t > 0$, and objective *strictly* increases.
  - So, cannot visit same BFS twice, and because finite number of BFS, simplex terminates. When it terminates the solution is optimal.

- Note: If LP is unbounded, at some point must find this during simplex.

Summary

- Tableau is system of eqns that correspond to basis, and represent all details of LP
  - Basic variables are isolated
  - Simplex changes tableau when searching BFS

- Simplex pivots through adjacent BFS until finds “unbounded,” or terminates with an optimal solution (will provably terminate if problem is non-degenerate).

- Simplex is fast in practice
Next lecture

• How to find a first BFS to initiate the method?
• How can we be sure the method terminates even if it encounters degeneracy?