# AM 121: Intro to Optimization Models and Methods

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## Lecture 5: The Simplex Method

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## Lesson Plan

This lecture: Moving towards an algorithm for solving LPs

- Tableau. Adjacent BFS.
- The **simplex method**

Jensen & Bard: 3.4, 3.5, 3.6, 3.10 (for matrix representation)

# Warm-up

| Canonical form | | Equivalent canonical form |
|---|---|---|
| max $c^T x$<br>s.t. $Ax=b$<br>$x \geq 0$ | $\Longleftrightarrow$ | max $\mathbf{z}$<br>s.t. $\mathbf{z - c^T x = 0}$<br>$Ax=b$<br>$x \geq 0$ |

•We're interested in constructing a **tableau** that corresponds to a basis B

•A tableau is a system of equations

•Represents **everything** about an LP

# Example

- max $x_1 + x_2$

  s.t.
  $$\begin{aligned} x_1 \quad\quad\quad + x_3 \quad\quad &= 2 \\ x_1 + 2x_2 \quad\quad + x_4 &= 4 \\ x_1, \; x_2, \; x_3, \; x_4 &\geq 0 \end{aligned}$$

- max $z$

  s.t.
  $$\begin{aligned} z - x_1 - x_2 \quad\quad\quad\quad &= 0 \\ x_1 \quad\quad\quad + x_3 \quad\quad &= 2 \\ x_1 + 2x_2 \quad\quad + x_4 &= 4 \\ x_1, \; x_2, \; x_3, \; x_4 &\geq 0 \end{aligned}$$

- The **tableau** corresponding to basis {3,4} is:

  $$\begin{aligned} z - x_1 - x_2 \quad\quad\quad\quad &= 0 \\ x_1 \quad\quad\quad + x_3 \quad\quad &= 2 \\ x_1 + 2x_2 \quad\quad + x_4 &= 4 \end{aligned}$$

  (everything about LP!)

# The Tableau

max z

s.t. $z - c^T x = 0$

$Ax=b$

$x \geq 0$

•**Definition.** The **tableau** for basis *B* is a system of eqns where the **basic variables are isolated**.

•For basis *B* (with *B'=N \ B*) the **tableau** is:

$$\begin{aligned} z \qquad\qquad + \bar{c}_{B'}^T x_{B'} &= \bar{v} \\ I x_B + \bar{A}_{B'} x_{B'} &= \bar{b} \end{aligned}$$

•Basic solution $x_B = \bar{b}$, $x_{B'} = 0$
•Objective $z = \bar{v}$
•Feasible when $\bar{b} \geq 0$
•Parameters $\bar{A}$, $\bar{c}$, $\bar{v}$, $\bar{b}$ different from those without bar
•RHS may be negative

Insisting on $x_{B'} = 0$ adds an additional n-m constraints. Pins down a solution to the LP!

# Constructing a Tableau

- Construct the tableau for basis B:

max z
s.t. $z - c^T x = 0$
$Ax=b$
$x \geq 0$

$\Longrightarrow$

$$\begin{aligned} z \qquad\qquad + \bar{c}_{B'}^T x_{B'} &= \bar{v} \\ I x_B + \bar{A}_{B'} x_{B'} &= \bar{b} \end{aligned}$$

a. Row operations to transform second row

$$A_B x_B + A_{B'} x_{B'} = b \quad \xRightarrow{\text{Multiply by } A_B^{-1}} \quad I x_B + A_B^{-1} A_{B'} x_{B'} = A_B^{-1} b \text{ (*)}$$

Let $\bar{A}_{B'} = A_B^{-1} A_{B'}$

and $\bar{b} = A_B^{-1} b$

# Constructing a Tableau

- Construct the tableau for basis B:

max z

s.t. z - c$^T$ x =0

Ax=b

x≥0

⟹

$$
\begin{aligned}
z \qquad\qquad + \bar{c}_{B'}^T x_{B'} &= \bar{v} \\
I x_B + \bar{A}_{B'} x_{B'} &= \bar{b}
\end{aligned}
$$

b. Row operations to transform first row

$$z - c_B^T x_B - c_{B'}^T x_{B'} = 0$$

# Constructing a Tableau

- Construct the tableau for basis B:

max z

s.t. z - c$^T$ x =0

Ax=b

x≥0

⟹

$$
\begin{aligned}
z \qquad\qquad + \bar{c}_{B'}^T x_{B'} &= \bar{v} \\
I x_B + \bar{A}_{B'} x_{B'} &= \bar{b}
\end{aligned}
$$

b. Row operations to transform first row

$$z - c_B^T x_B - c_{B'}^T x_{B'} = 0$$

$\oplus$ $$x_B + A_B^{-1} A_{B'} x_{B'} = A_B^{-1} b \qquad (*)$$

# Constructing a Tableau

- Construct the tableau for basis B:

| max z |
|---|
| s.t. z - $c^T$ x =0 |
| Ax=b |
| x≥0 |

$$\Longrightarrow \quad \begin{aligned} z \quad\quad\quad + \bar{c}_{B'}^T x_{B'} &= \bar{v} \\ I x_B + \bar{A}_{B'} x_{B'} &= \bar{b} \end{aligned}$$

b. Row operations to transform first row

$$z - c_B^T x_B - c_{B'}^T x_{B'} = 0$$

$$\oplus \quad c_B^T (x_B + A_B^{-1} A_{B'} x_{B'}) = c_B^T A_B^{-1} b \qquad c_B^T(*)$$


# Constructing a Tableau

- Construct the tableau for basis B:

| max z |
|---|
| s.t. z - $c^T$ x =0 |
| Ax=b |
| x≥0 |

$$\Longrightarrow \quad \begin{aligned} z \quad\quad\quad + \bar{c}_{B'}^T x_{B'} &= \bar{v} \\ I x_B + \bar{A}_{B'} x_{B'} &= \bar{b} \end{aligned}$$

b. Row operations to transform first row

$$z - c_B^T x_B - c_{B'}^T x_{B'} = 0$$

$$\oplus \quad c_B^T (x_B + A_B^{-1} A_{B'} x_{B'}) = c_B^T A_B^{-1} b \qquad c_B^T(*)$$


$$z + c_B^T A_B^{-1} A_{B'} x_{B'} - c_{B'}^T x_{B'} = c_B^T A_B^{-1} b$$

# Constructing a Tableau

- Construct the tableau for basis B:

| |
|---|
| max z |
| s.t. $z - c^T x = 0$ |
| $Ax=b$ |
| $x \geq 0$ |

$$\Rightarrow \quad \begin{aligned} z \quad\quad &+ \bar{c}_{B'}^T x_{B'} = \bar{v} \\ I x_B &+ \bar{A}_{B'} x_{B'} = \bar{b} \end{aligned}$$

b. Row operations to transform first row

$$z - c_B^T x_B - c_{B'}^T x_{B'} = 0$$

$$\oplus \quad c_B^T (x_B + A_B^{-1} A_{B'} x_{B'}) = c_B^T A_B^{-1} b \qquad c_B^T(*)$$

$$z + c_B^T A_B^{-1} A_{B'} x_{B'} - c_{B'}^T x_{B'} = c_B^T A_B^{-1} b$$

Let $\bar{c}_{B'}^T = c_B^T A_B^{-1} A_{B'} - c_{B'}^T$ and
$\bar{v} = c_B^T A_B^{-1} b$

# Example: Tableau (1 of 3)

- $z - x_1 - x_2 = 0$ (a) $\quad$ x=(0,0,2,4) B={3,4}
  $x_1 + x_3 = 2$ (b)
  $x_1 + 2x_2 + x_4 = 4$ (c)
- How obtain tableau for B={1,4}?
- Subtract (b) from (c); add (b) to (a):
- $z - x_2 + x_3 = 2$ $\quad$ x=(2,0,0,2) B={1,4}
  $x_1 + x_3 = 2$
  $2x_2 - x_3 + x_4 = 2$
- BFS? Objective?
- x=(2,0,0,2) ; z=2. **An improvement!**

## Example: Tableau (2 of 3)

- $z \quad - x_2 \quad + x_3 \quad = 2$
  $x_1 \quad + x_3 \quad = 2$
  $2x_2 \quad - x_3 \quad + x_4 = 2$

x=(2,0,0,2)
B={1,4}

**How can we improve further?** *Have 2 enter basis. What leaves?*

- Increase $x_2$:=t; keep $x_3$=0; z= 2+t
- Feasibility: $x_4$=2 – 2t.
- Set $x_2$:=1, because $x_4$<0 for t>1.
- New basis B={1,2}. x=(2,1,0,0), z=3.

## Example: Tableau (3 of 3)

- $z \quad - x_2 \quad + x_3 \quad = 2$ (a)
  $x_1 \quad + x_3 \quad = 2$ (b)
  $2x_2 \quad - x_3 \quad + x_4 = 2$ (c)

x=(2,0,0,2)
B={1,4}

Get tableau for B={1,2}

- $z + \quad + \frac{1}{2} x_3 \quad + \frac{1}{2} x_4 = 3$
  $x_1 \quad + \quad x_3 \quad = 2$
  $x_2 - \frac{1}{2} x_3 \quad + \frac{1}{2} x_4 = 1$

x=(2,1,0,0)
B={1,2}

- Obj coeff of nonbasic vars >= 0. Solution optimal!
- Must have $z = 3 - \frac{1}{2} x_3 - \frac{1}{2} x_4 \leq 3$ for any feasible solution, since $x_3, x_4 \geq 0$.

# From Basis to Tableau and Back

- **Theorem**. If $B$ is a basis for matrix $\boldsymbol{A}$ then there is a tableau for $B$. If there is a tableau for $B$ then $B$ is a basis for $\boldsymbol{A}$.
- **Proof**

  (➔) row operations to isolate the variables $\boldsymbol{x}_B$ (possible since $\boldsymbol{A}_B$ is invertible)

  (🡐) $B$ is a basis because $\boldsymbol{A}_B$=**I,** and therefore $\boldsymbol{A}_B$ spans.

# Adjacent Bases

- **Definition.** Two bases for matrix $\boldsymbol{A}$ are **adjacent** if they share all but one column
- **Example**
  - max $z = x_1 + x_2$
  - s.t.

$$
\begin{aligned}
x_1 \quad\quad\;\; + x_3 \quad\quad &= 2 \\
x_1 + 2x_2 \quad\quad + x_4 &= 4 \\
x_1, \quad x_2, \quad x_3, \quad x_4 &\geq 0
\end{aligned}
$$

- Five bases {1,2}, {1,3}, {1,4}, {2,3}, {3,4}.
- B={1,2} is adjacent to {1,3}, not {3,4}.

## The Simplex Method

Work in tableau form.

- Step **(i)**. Find an initial BFS. ⟵ **Next lecture**
- Step **(ii)**.
  - – Determine if current BFS is optimal.
  - – If not, find an adjacent BFS with a larger z-value.
- Repeat.

## How fast is this?

- Number of possible BFS is "n choose m"

  $^{n}C_{m}$ = n! / (n-m)! m!
- $^{20}C_{10}$=184,756
- Even if we can avoid cycling… gets large.
  - – "degeneracy" will lead to concern about cycling (next lecture)

## Empirical performance

- Applied to LPs in canonical form, tends to find an optimal solution after searching less than **3m BFS.**
- Example: 20-variable, 10-constraint problem: search ≈ 30 BFS.
- Simplex has very good performance in practice

## Simplex Method - Overview

- Canonical form: max $z$

  s.t. $z - c^T x = 0$

  $Ax=b,\ x \geq 0$
- Step 0. Initialization
- Step 1. Check optimality.
- Step 2. Choose entering index.
- Step 3. Check unboundedness.
- Step 4. Choose leaving index.
- Step 5. Pivot to a new tableau. **To step 1**.

# Simplex Method - Overview

- max $z$
  s.t. $z \quad + \sum_{j \in B'} \bar{c}_j x_j = \bar{v}$
  $x_i \quad + \sum_{j \in B'} \bar{a}_{ij} x_j = \bar{b}_i \quad (i \in B)$ (assume $\bar{b}_i \geq 0$)
- **Step 0**. Initialization (find a BFS and tableau)
- **Step 1**. If $\bar{c}_j \geq 0$ for all $j \in B'$, *stop* (optimal)
- **Step 2**. Else, pick entering index $k \in B'$ with $\bar{c}_k < 0$.
- **Step 3**. If $\bar{a}_{ik} \leq 0$ for all $i \in B$, *stop* (unbounded)
- **Step 4**. **Ratio test**. $t^* = \min\{ \bar{b}_i / \bar{a}_{ik} : i \in B, \bar{a}_{ik} > 0\}$. Pick *leaving index* $r \in B$ with min ratio.
- **Step 5**. **Pivot** on $(r,k)$ entry to get a tableau for $B := B \cup \{k\} \setminus \{r\}$.

# Example of Simplex Method

$$
\begin{aligned}
z - x_1 - x_2 &= 0 \\
\leftarrow \quad \boxed{x_1} \quad + x_3 &= 2 \\
x_1 + 2x_2 \quad + x_4 &= 4 \\
\uparrow \quad &
\end{aligned}
$$

Basic $x_3$ $x_4$
Ratio 2/1 4/1
$x_1$ to enter. $x_3$ to leave. pivot(3,1)

$$
\begin{aligned}
z \quad - x_2 + x_3 &= 2 \\
x_1 \quad + x_3 &= 2 \\
\leftarrow \quad \boxed{2x_2} - x_3 + x_4 &= 2 \\
\uparrow \quad &
\end{aligned}
$$

Basic $x_1$ $x_4$
Ratio 2/2
$x_2$ to enter. $x_4$ to leave. pivot(4,2)

$$
\begin{aligned}
z \quad + \tfrac{1}{2} x_3 + \tfrac{1}{2} x_4 &= 3 \\
x_1 \quad + x_3 &= 2 \\
x_2 - \tfrac{1}{2} x_3 + \tfrac{1}{2} x_4 &= 1
\end{aligned}
$$

Basic $x_1$ $x_2$
Reduced costs all nonnegative.
**Stop!**

- Solution: $(x_1,x_2,x_3,x_4)=(2,1,0,0)$, $z=3$.

## Review

## Step 0: Initialization

- Find a basis and initial tableau corresponding to a BFS
  - next lecture

## Step 1: Check optimality

- If $\bar{c}_j \geq 0$ (for all j in B') then **stop, current solution is optimal**.
- Value z = $\bar{v}$. Any feasible solution satisfies

$$z = \bar{v} - \bar{c}^T x_{B'} \leq \bar{v},$$

since $\bar{c} \geq 0$ and $x \geq 0$.

- Non-negative reduced costs is a sufficient test for optimality. [It is a valid test even if a solution is "degenerate", discussed in next lecture.]

## Step 2: Choose Entering Index

- Pick some k in B' with $\bar{c}_k<0$ to enter the basis, so that the obj value increases with $x_k$

- Say that $\bar{c}_k$ is the "**reduced cost"** of nonbasic variable $x_k$. Amount by which z *decreases* when $x_k$ increases (and so $\bar{c}_k<0$ is good).

## Step 3: Check Unboundedness

- $x_i + \sum_{j \in B'} \bar{a}_{ij} x_j = \bar{b}_i$ (for all $i \in B$)
- Because other nonbasic vars = 0, we can increase $x_k$ while:

  $x_i = \bar{b}_i - \bar{a}_{ik} x_k \geq 0$ (for all $i \in B$)
- If $\bar{a}_{ik} \leq 0$ for every i in B, then $x_k$ can increase without bound (since $x_i$ can increase without affecting objective)!
- **Stop: objective value is unbounded.**

## Step 4: Choose Leaving Index

- Need: $x_i = \bar{b}_i - \bar{a}_{ik} x_k \geq 0$; and so $x_k \leq \bar{b}_i / \bar{a}_{ik}$ (where $\bar{a}_{ik} > 0$).
- **Ratio test:** $t^* = \min\{ \bar{b}_i / \bar{a}_{ik} : i \text{ in } B, \bar{a}_{ik} > 0\}$
- Let *R* denote basic vars with min ratio
- Pick some r in *R* as leaving index; set $x_{\mathbf{k}} := t^*$ (may be zero)
- New basic variable $x_{\mathbf{k}}$ "**k**icks out" the "**r**emoved" variable $x_{\mathbf{r}}$

# Step 5: Pivot to new tableau

- **Definition**. A **pivot** on (r,k) is row operations to construct tableau for $B:=B\cup\{k\}\setminus\{r\}$.

- **(a)** Divide row $x_r + \sum_{j\in B'} \bar{a}_{rj}x_r = \bar{b}_r$ through by $\bar{a}_{rk}$ so that coefficient of new basic variable $x_k$ becomes 1.

  *(Why does RHS remain nonnegative?)*

- **(b)** Add/subtract multiples of this adjusted row to all other equations (including objective) to remove $x_k$

  *(Why do these operations not affect coeffs for other basic vars?)*

[Note: we're doing "Gauss-Jordan elimination."]

# Degeneracy

- A BFS is "**degenerate**" if one or more basic variables have value zero.
- In this case (unless unbounded), $t^* = 0$ and simplex method cannot make the entering variable increase in value.
- Simplex will move to an adjacent basis, but without improving the objective.
- For now we ignore this possibility.

## Summary: Simplex Method

- A tableau is system of eqns that correspond to a basis and represent **all** details of the LP
  - the basic variables are isolated
- The simplex method changes the tableau when searching through BFS.
- Pivots through adjacent BFSs until prove LP is unbounded or find an optimal solution.
- Fast in practice!

## Next lecture

- How to initialize the simplex method?
- Provable termination even when simplex method encounters a degenerate BFS?