AM 121: Intro to Optimization Models and Methods

Fall 2017

Lecture 3: Applications, Examples, Exercises.

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Lecture 3: Lesson plan

- The Post Office Problem
- The SailCo Problem
- The SAVE-IT Company

- A Simple AMPL Example
  (AMPL: A Modeling Language for Mathematical Programming)

- From problem to LP (+ a little AMPL)

Post Office Problem

Union rules state that each full-time employee must work 5 consecutive days and then receive 2 days off.

Formulate an LP to minimize number of full-time employees who must be hired.

(Assume solution will be integral)

Day	Number of full-time employees required
1=Monday	17
2=Tuesday	13
3=Wednesday	15
4=Thursday	19
5=Friday	14
6=Saturday	16
7=Sunday	11

$$
\begin{aligned}
\min z = \sum_i x_i & \\
\text{s.t.} \quad \sum_i x_i - x_2 - x_3 &\geq 17 \\
\sum_i x_i - x_3 - x_4 &\geq 13 \\
\sum_i x_i - x_4 - x_5 &\geq 15 \\
\sum_i x_i - x_5 - x_6 &\geq 19 \\
\sum_i x_i - x_6 - x_7 &\geq 14 \\
\sum_i x_i - x_7 - x_1 &\geq 16 \\
\sum_i x_i - x_1 - x_2 &\geq 11 \\
x_i &\geq 0
\end{aligned}
$$

x_i: number of people with shift start on day i

Note: will need solution to be integral!

Variation 1: Forced overtime

- Post office can ask employees to work a 6th day each week (i.e. work for 6 consecutive days)
- Pay \$500/wk, \$130 for the overtime day.
- Formulate an LP to minimize weekly labor costs
- (Assume solution will be integral)

Day	Number of full-time employees required
1=Monday	17
2=Tuesday	13
3=Wednesday	15
4=Thursday	19
5=Friday	14
6=Saturday	16
7=Sunday	11

$$\min z = 500 \sum_i x_i + 130 \sum_i o_i$$

$$\begin{aligned}
\text{s.t.} \quad & \sum_i x_i - x_2 - x_3 + o_3 \geq 17 \\
& \sum_i x_i - x_3 - x_4 + o_4 \geq 13 \\
& \sum_i x_i - x_4 - x_5 + o_5 \geq 15 \\
& \sum_i x_i - x_5 - x_6 + o_6 \geq 19 \\
& \sum_i x_i - x_6 - x_7 + o_7 \geq 14 \\
& \sum_i x_i - x_7 - x_1 + o_1 \geq 16 \\
& \sum_i x_i - x_1 - x_2 + o_2 \geq 11 \\
& o_i \leq x_i, \quad \forall i \\
& x_i, o_i \geq 0, \quad \forall i
\end{aligned}$$

Note: will need solution to be integral!

x_i: number of people with shift start on day i

o_i: number of people who work one day overtime who start on day i

Variation 2: Maximizing Weekends!

- Post office has 25 full-time employees. Cannot hire or fire.

Day	Number of full-time employees required
1=Monday	17
2=Tuesday	13
3=Wednesday	15
4=Thursday	19
5=Friday	14
6=Saturday	16
7=Sunday	11

- Formulate an LP to schedule employees to maximize the number of weekend days off
- (Assume solution will be integral)

$$\begin{aligned}
\max z &= x_7 + 2x_1 + x_2 \\
\text{s.t.} \quad & \sum_i x_i - x_2 - x_3 \geq 17 \\
& \sum_i x_i - x_3 - x_4 \geq 13 \\
& \sum_i x_i - x_4 - x_5 \geq 15 \\
& \sum_i x_i - x_5 - x_6 \geq 19 \\
& \sum_i x_i - x_6 - x_7 \geq 14 \\
& \sum_i x_i - x_7 - x_1 \geq 16 \\
& \sum_i x_i - x_1 - x_2 \geq 11 \\
& \sum_i x_i = 25 \\
& x_i \geq 0
\end{aligned}$$

x_i : number of people who start shift on day i

Note: will need solution to be integral!

Variation 3:Part-time Employees

- Can hire part-time employees
- Full-time: 8 hours a day, 5 consecutive days (2 days off). Cost \$15/hour.
- Part-time: 4 hours a day, 5 consecutive days (2 days off). Cost \$10/hour.
- Part-time limited by union to fill 25% of weekly labor (in terms of working hours).
- Formulate an LP to minimize weekly labor costs. (Assume solution will be integral)

Day	Number of hours required
1=Monday	8*17=136 hrs
2=Tuesday	8*13=104 hrs
3=Wednesday	8*15=120 hrs
4=Thursday	8*19=152 hrs
5=Friday	8*14=112 hrs
6=Saturday	8*16=128 hrs
7=Sunday	8*11=88 hrs

x_i: number full time who start shift on day i

p_i: number part-time who start shift on day i

$$\min z = 120\sum_i x_i + 40\sum_i p_i$$

$$\begin{aligned}
\text{s.t.}\quad & 8(\sum_i x_i - x_2 - x_3) + 4(\sum_i p_i - p_2 - p_3) \geq 136 \\
& 8(\sum_i x_i - x_3 - x_4) + 4(\sum_i p_i - p_3 - p_4) \geq 104 \\
& 8(\sum_i x_i - x_4 - x_5) + 4(\sum_i p_i - p_4 - p_5) \geq 120 \\
& 8(\sum_i x_i - x_5 - x_6) + 4(\sum_i p_i - p_5 - p_6) \geq 152 \\
& 8(\sum_i x_i - x_6 - x_7) + 4(\sum_i p_i - p_6 - p_7) \geq 112 \\
& 8(\sum_i x_i - x_7 - x_1) + 4(\sum_i p_i - p_7 - p_1) \geq 128 \\
& 8(\sum_i x_i - x_1 - x_2) + 4(\sum_i p_i - p_1 - p_2) \geq 88
\end{aligned}$$

Note: will need solution to be integral!

$$4\sum_i p_i \leq 0.25(4\sum_i p_i + 8\sum_i x_i)$$

$$x_i, p_i \geq 0$$

The SailCo Problem

SailCo must determine how many sailboats to produce in each quarter in order to meet demand. Boats can be used in same quarter produced, and held-over to future quarter.

Formulate an LP to determine a production schedule for Q1-Q4 to min total production and inventory costs. (Assume there is no demand after Q4.)

	Q1	Q2	Q3	Q4
Forecast demand	40	60	75	25

Production cost

$400/boat, first 40 boats in a quarter

$450/boat for additional boats

Inventory cost

$20/boat/quarter for boats on hand at end of a quarter (after production has occurred and demand satisfied)

Initial inventory: 10 sailboats at start of Q1

$$
\begin{aligned}
\min z = 400 \sum_t x_t + 450 \sum_t y_t + 20 \sum_t h_t \\
\text{s.t.} \quad x_t \leq 40, \quad \forall t \\
h_1 = 10 + x_1 + y_1 - 40 \\
h_2 = h_1 + x_2 + y_2 - 60 \\
h_3 = h_2 + x_3 + y_3 - 75 \\
h_4 = h_3 + x_4 + y_4 - 25 \\
h_t, y_t, x_t \geq 0
\end{aligned}
$$

h_t: represents # boats on hand at end of quarter
x_t: number of boats made up to 40
y_t: number of boats made above 40

Variation 1: A Rolling Horizon

- Suppose make 40 in Q1, and actual demand is 35.
- Start Q2 with 10+40-35=15 boats on hand.
- New *planning period* is Q2-Q5.
- Currently the optimal solution to the LP will not try to keep boats on hand at end of Q5.
- Modify the formulation to work for Q2-Q5, and ensure that we end the Q5 "planning horizon" with 10 boats in inventory.
- Forecast for Q2-Q5:

	Q1	Q2	Q3	Q4	Q5
forecast demand	40	60	75	25	36
actual demand	35				

Formulate an LP for Q2-Q5 to determine a production schedule to minimize sum of costs and meet new constraint.

Formulation for Q2-Q5:

$$\min z = 400 \sum_t x_t + 450 \sum_t y_t + 20 \sum_t h_t$$

$$\begin{aligned} \text{s.t.} \quad & x_t \le 40, \quad \forall t \\ & h_2 = 15 + x_2 + y_2 - 60 \\ & h_3 = h_2 + x_3 + y_3 - 75 \\ & h_4 = h_3 + x_4 + y_4 - 25 \\ & h_5 = h_4 + x_5 + y_5 - 36 \\ & h_5 \ge 10 \\ h_t, y_t, x_t \ge 0 & \end{aligned}$$

h_t: represents # boats on hand at end of quarter
x_t: number of boats made up to 40
y_t: number of boats made above 40
decision variables for periods 2, …, 5

Variation 2: Production Smoothing

- Make production "smooth" across periods:
  - an increase in production costs $400/boat (training)
  - a decrease in production costs $500/boat (severance pay, loss in morale)
- Assume 50 boats made during the Q preceding Q1, initial inventory of 10.
- Need at least 10 boats on hand at end of planning horizon

	Q1	Q2	Q3	Q4
Forecast demand	40	60	75	25

- Formulate an LP for Q1-Q4 to determine a production schedule to minimize sum of production and inventory costs.

$$\min z = 400\sum_t x_t + 450\sum_t y_t + 20\sum_t h_t + 400\sum_t c_t^+ + 500\sum_t c_t^-$$

$$\begin{aligned}
\text{s.t.}\quad & x_t \leq 40, \quad \forall t \\
& h_1 = 10 + x_1 + y_1 - 40 \\
& h_2 = h_1 + x_2 + y_2 - 60 \\
& h_3 = h_2 + x_3 + y_3 - 75 \\
& h_4 = h_3 + x_4 + y_4 - 25 \\
& h_4 \geq 10
\end{aligned}$$

$$\begin{aligned}
x_1 + y_1 - 50 &= c_1^+ - c_1^- \\
x_2 + y_2 - (x_1 + y_1) &= c_2^+ - c_2^- \\
x_3 + y_3 - (x_2 + y_2) &= c_3^+ - c_3^- \\
x_4 + y_4 - (x_3 + y_3) &= c_4^+ - c_4^- \\
h_t, y_t, x_t, c_t^+, c_t^- &\geq 0
\end{aligned}$$

Solution: x=(40,40,40,40); y=(15,15,15,15)

h_t: # boats on hand at end of quarter
x_t, y_t: number of boats made up to (above) 40
c_t+ : #boat increase from last period
c_t- : #boat decrease from last period

- Why will the optimal solution not have both $c_t^+ > 0$ and $c_t^- > 0$?
- Suppose production 60 in period 1 and 70 in period 2. What assignments to c_2^+ and c_2^- are *feasible*?

$$70 - 60 = c_2^+ - c_2^-$$
$$c_2^+, c_2^- \geq 0$$

- What assignments are *optimal*?

$$\min \quad \ldots + \; 400c_2^+ + 500c_2^- + \ldots$$

Variation 3: Allowing demands to be backlogged

- Suppose demands can be met in future periods.
- Penalty $100/boat per quarter demand is unmet.
- Must meet all demand by end of Q4.
- Build from variation 2 (still face smoothing costs with 50 in previous period and 10 on hand at start of Q1; still want 10 on hand at end of Q4.)

	Q1	Q2	Q3	Q4
Forecast demand	40	60	75	25

Formulate an LP to determine a production schedule to minimize the sum of the production and inventory costs.

$$
\begin{aligned}
\min z = 400\sum_t x_t + 450\sum_t y_t + 20\sum_t h_t^+ + 400\sum_t c_t^+ + 500\sum_t c_t^- \\
+ 100\sum_t h_t^-
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t.}\quad x_t \le 40, \quad \forall t \\
h_1^+ - h_1^- &= 10 + x_1 + y_1 - 40 \\
h_2^+ - h_2^- &= h_1^+ - h_1^- + x_2 + y_2 - 60 \\
h_3^+ - h_3^- &= h_2^+ - h_2^- + x_3 + y_3 - 75 \\
h_4^+ - h_4^- &= h_3^+ - h_3^- + x_4 + y_4 - 25 \\
x_1 + y_1 - 50 &= c_1^+ - c_1^- \\
x_2 + y_2 - (x_1 + y_1) &= c_2^+ - c_2^- \\
x_3 + y_3 - (x_2 + y_2) &= c_3^+ - c_3^- \\
x_4 + y_4 - (x_3 + y_3) &= c_4^+ - c_4^- \\
h_4^+ &\ge 10 \\
h_4^- &\le 0 \\
h_t^+, h_t^-, y_t, x_t, c_t^+, c_t^- &\ge 0
\end{aligned}
$$

h_t^+ : #boats on hand at end Q
H_t^- : #boats backlogged at end Q

The SAVE-IT Company

- Operates a *recycling center*. Collects four types *materials*, treats, and amalgamates to make an insulation product of three different *grades*.
- Formulate an LP to determine the amount of each grade and the mix of materials for each grade, maximizing profit (sales – cost)

Have grant of $30,000/wk for treatment cost (can't treat more than this)

At least half of each type of material must be treated.

Material	Availability (lbs /wk)	Treatment cost ($/lb)
1	3000	3
2	2000	6
3	4000	4
4	1000	5

Grade	Spec	Amalg. cost ($/lb)	Sales price ($/lb)
A	1 <= 30% 2 >= 40 % 3 <= 50% 4 =20%	3	8.5
B	1 <= 50% 2 >= 10% 3: any 4 = 10%	2.5	7
C	1 <= 70% 2,3,4: any	2	5.5

$$
\begin{aligned}
\max 5.5 \sum_j x_{Aj} + 4.5 \sum_j x_{Bj} + 3.5 \sum_j x_{Cj} & \\
\text{s.t.} \quad x_{A1} \le 0.3 \sum_j x_{Aj} & \\
x_{A2} \ge 0.4 \sum_j x_{Aj} & \\
x_{A3} \le 0.5 \sum_j x_{Aj} & \\
x_{A4} = 0.2 \sum_j x_{Aj} & \\
x_{B1} \le 0.5 \sum_j x_{Bj} & \\
x_{B2} \ge 0.1 \sum_j x_{Bj} & \\
x_{B4} = 0.1 \sum_j x_{Bj} & \\
x_{C1} \le 0.7 \sum_j x_{Cj} & \\
1500 \le x_{A1} + x_{B1} + x_{C1} \le 3000 & \\
1000 \le x_{A2} + x_{B2} + x_{C2} \le 2000 & \\
2000 \le x_{A3} + x_{B3} + x_{C3} \le 4000 & \\
500 \le x_{A4} + x_{B4} + x_{C4} \le 1000 & \\
3(x_{A1} + x_{B1} + x_{C1}) + 6(x_{A2} + x_{B2} + x_{C2}) & \\
+4(x_{A3} + x_{B3} + x_{C3}) + 5(x_{A4} + x_{B4} + x_{C4}) \le 30000 & \quad (*) \\
x_{A1}, x_{A2}, \ldots, x_{C4} \ge 0 &
\end{aligned}
$$

x_{A2}: quantity of material type 2 used for product A

Variation 1: Pay for Treatment

- SAVE-IT recognizes that it might be able to to make more profits and improve the environment.
- Gets matching funds: provide a 80% rebate on treatment cost above $30,000 per week.
- Formulate an LP that (a) uses all of $30,000, (b) treats at least half of each type of material, (c) may treat more material if this is profitable.

- Introduce new constraint:

$$y = 3(x_{A1} + x_{B1} + x_{C1}) + 6(x_{A2} + x_{B2} + x_{C2}) + 4(x_{A3} + x_{B3} + x_{C3}) + 5(x_{A4} + x_{B4} + x_{C4}) - 30000$$

- Decision variable (total dollar cost of excess treatment) $y \geq 0$
- Add $-0.2\,y$ to the objective
- Drop inequality (*)

- The new formulation is on the next slide

$$\begin{aligned}
\max\ & 5.5 \sum_j x_{Aj} + 4.5 \sum_j x_{Bj} + 3.5 \sum_j x_{Cj} - 0.2y \\
\text{s.t.}\quad & x_{A1} \leq 0.3 \sum_j x_{Aj} \\
& x_{A2} \geq 0.4 \sum_j x_{Aj} \\
& x_{A3} \leq 0.5 \sum_j x_{Aj} \\
& x_{A4} = 0.2 \sum_j x_{Aj} \\
& x_{B1} \leq 0.5 \sum_j x_{Bj} \\
& x_{B2} \geq 0.1 \sum_j x_{Bj} \\
& x_{B4} = 0.1 \sum_j x_{Bj} \\
& x_{C1} \leq 0.7 \sum_j x_{Cj} \\
& 1500 \leq x_{A1} + x_{B1} + x_{C1} \leq 3000 \\
& 1000 \leq x_{A2} + x_{B2} + x_{C2} \leq 2000 \\
& 2000 \leq x_{A3} + x_{B3} + x_{C3} \leq 4000 \\
& 500 \leq x_{A4} + x_{B4} + x_{C4} \leq 1000 \\
& y = 3(x_{A1} + x_{B1} + x_{C1}) + 6(x_{A2} + x_{B2} + x_{C2}) \\
& \quad +4(x_{A3} + x_{B3} + x_{C3}) + 5(x_{A4} + x_{B4} + x_{C4}) - 30000 \\
& y, x_{A1}, x_{A2}, \ldots, x_{C4} \geq 0
\end{aligned}$$

A Simple AMPL Example

	Product A	Product B
Profit Per Unit	5	8
Machine time	1	1
Storage Space	5	9

$$\begin{aligned} \max \quad & 5x_1 + 8x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 6 \\ & 5x_1 + 9x_2 \le 45 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Total machine time = 6; Total storage space = 45;

example.mod

```
set PRODUCT;
set RESOURCE;
param usage {i in RESOURCE, j in PRODUCT};
param profit {j in PRODUCT};
param avail {i in RESOURCE};
var X {j in PRODUCT}>=0;

maximize Total_Profit: sum {j in PRODUCT} profit[j]*X[j];

subject to Resource_Constraints {i in RESOURCE}:
        sum{j in PRODUCT} usage[i, j]*X[j]<=avail[i];
```

example.dat

```
set PRODUCT := A  B;
set RESOURCE := machine space;

param usage:   A      B:=
      machine  1      1
      space    5      9;
param profit:= A  5   B   8;
param: avail:= machine 6  space 45;
```

Formulate in AMPL, solve with CPLEX

- Install AMPL
- Start AMPL using AMPL IDE or at command line

ampl: **model example.mod;**

ampl: **data example.dat;**

ampl: **option solver cplex;**

ampl: **solve;**

CPLEX 12.5.0.0: optimal solution; objective 41.25

2 dual simplex iterations (1 in phase I)

ampl: **display X;**

X [*] :=

A 2.25

B 3.75

;