AM 121: Intro to Optimization Models and Methods
Fall 2018

Lecture 3: Applications, Examples, Exercises.

Lecture 3: Lesson plan

• The Post Office Problem
• The SailCo Problem
• The SAVE-IT Company

• A Simple AMPL Example
  (AMPL: A Modeling Language for Mathematical Programming)

• From problem to LP (+ a little AMPL)
Post Office Problem

Union rules state that each full-time employee must work 5 consecutive days and then receive 2 days off.

**Day** | **Number of full-time employees required**
--- | ---
1=Monday | 17
2=Tuesday | 13
3=Wednesday | 15
4=Thursday | 19
5=Friday | 14
6=Saturday | 16
7=Sunday | 11

Formulate an LP to minimize number of full-time employees who must be hired.

(Assume solution will be integral)

\[
\begin{align*}
\text{min } z &= \sum_i x_i \\
\text{s.t. } & \sum_i x_i - x_2 - x_3 \geq 17 \\
& \sum_i x_i - x_3 - x_4 \geq 13 \\
& \sum_i x_i - x_4 - x_5 \geq 15 \\
& \sum_i x_i - x_5 - x_6 \geq 19 \\
& \sum_i x_i - x_6 - x_7 \geq 14 \\
& \sum_i x_i - x_7 - x_1 \geq 16 \\
& \sum_i x_i - x_1 - x_2 \geq 11 \\
& x_i \geq 0
\end{align*}
\]

x_i: number of people with shift start on day i

Note: will need solution to be integral!
Variation 1: Forced overtime

- Post office can ask employees to work a 6\textsuperscript{th} day each week (i.e. work for 6 consecutive days)
- Pay $500/wk, $130 for the overtime day.
- Formulate an LP to minimize weekly labor costs
- (Assume solution will be integral)

<table>
<thead>
<tr>
<th>Day</th>
<th>Number of full-time employees required</th>
</tr>
</thead>
<tbody>
<tr>
<td>1=M</td>
<td>17</td>
</tr>
<tr>
<td>2=T</td>
<td>13</td>
</tr>
<tr>
<td>3=W</td>
<td>15</td>
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<tr>
<td>4=T</td>
<td>19</td>
</tr>
<tr>
<td>5=F</td>
<td>14</td>
</tr>
<tr>
<td>6=S</td>
<td>16</td>
</tr>
<tr>
<td>7=S</td>
<td>11</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{min } z &= 500 \sum_i x_i + 130 \sum_i o_i \\
\text{s.t. } & \sum_i x_i - x_2 - x_3 + o_3 \geq 17 \\
& \sum_i x_i - x_3 - x_4 + o_4 \geq 13 \\
& \sum_i x_i - x_4 - x_5 + o_5 \geq 15 \\
& \sum_i x_i - x_5 - x_6 + o_6 \geq 19 \\
& \sum_i x_i - x_6 - x_7 + o_7 \geq 14 \\
& \sum_i x_i - x_7 - x_1 + o_1 \geq 16 \\
& \sum_i x_i - x_1 - x_2 + o_2 \geq 11 \\
\end{align*}
\]

\(x_i\): number of people with shift start on day \(i\)
\(o_i\): number of people who work one day overtime who start on day \(i\)

Note: will need solution to be integral!

\[
\begin{align*}
o_i & \leq x_i, \quad \forall i \\
x_i, o_i & \geq 0, \quad \forall i
\end{align*}
\]
Variation 2: Maximizing Weekends!

- Post office has 25 full-time employees. Cannot hire or fire.

- Formulate an LP to schedule employees to maximize the number of weekend days off
- (Assume solution will be integral)

<table>
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<tr>
<td>1=Monday</td>
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</tr>
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</tr>
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</tr>
<tr>
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<td>19</td>
</tr>
<tr>
<td>5=Friday</td>
<td>14</td>
</tr>
<tr>
<td>6=Saturday</td>
<td>16</td>
</tr>
<tr>
<td>7=Sunday</td>
<td>11</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{max } z &= x_7 + 2x_1 + x_2 \\
\text{s.t. } &
\sum_i x_i - x_2 - x_3 \geq 17 \\
&
\sum_i x_1 - x_3 - x_4 \geq 13 \\
&
\sum_i x_1 - x_4 - x_5 \geq 15 \\
&
\sum_i x_1 - x_5 - x_6 \geq 19 \\
&
\sum_i x_1 - x_6 - x_7 \geq 14 \\
&
\sum_i x_1 - x_7 - x_1 \geq 16 \\
&
\sum_i x_1 - x_1 - x_2 \geq 11 \\
&
\sum_i x_i = 25 \\
x_i &\geq 0
\end{align*}
\]

Note: will need solution to be integral!
Variation 3: Part-time Employees

- Can hire part-time employees
- Full-time: 8 hours a day, 5 consecutive days (2 days off). Cost $15/hour.
- Part-time: 4 hours a day, 5 consecutive days (2 days off). Cost $10/hour.
- Part-time limited by union to fill 25% of weekly labor (in terms of working hours).
- Formulate an LP to minimize weekly labor costs. (Assume solution will be integral)

<table>
<thead>
<tr>
<th>Day</th>
<th>Number of hours required</th>
</tr>
</thead>
<tbody>
<tr>
<td>1=Monday</td>
<td>8*17=136 hrs</td>
</tr>
<tr>
<td>2=Tuesday</td>
<td>8*13=104 hrs</td>
</tr>
<tr>
<td>3=Wednesday</td>
<td>8*15=120 hrs</td>
</tr>
<tr>
<td>4=Thursday</td>
<td>8*19=152 hrs</td>
</tr>
<tr>
<td>5=Friday</td>
<td>8*14=112 hrs</td>
</tr>
<tr>
<td>6=Saturday</td>
<td>8*16=128 hrs</td>
</tr>
<tr>
<td>7=Sunday</td>
<td>8*11=88 hrs</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\min z &= 120 \sum_i x_i + 40 \sum_i p_i \\
\text{s.t.} & \\
8(\sum_i x_i - x_2 - x_3) + 4(\sum_i p_i - p_2 - p_3) & \geq 136 \\
8(\sum_i x_i - x_3 - x_4) + 4(\sum_i p_i - p_3 - p_4) & \geq 104 \\
8(\sum_i x_i - x_4 - x_5) + 4(\sum_i p_i - p_4 - p_5) & \geq 120 \\
8(\sum_i x_i - x_5 - x_6) + 4(\sum_i p_i - p_5 - p_6) & \geq 152 \\
8(\sum_i x_i - x_6 - x_7) + 4(\sum_i p_i - p_6 - p_7) & \geq 112 \\
8(\sum_i x_i - x_7 - x_1) + 4(\sum_i p_i - p_7 - p_1) & \geq 128 \\
8(\sum_i x_i - x_1 - x_2) + 4(\sum_i p_i - p_1 - p_2) & \geq 88 \\
4 \sum_i p_i & \leq 0.25(4 \sum_i x_i + 8 \sum_i x_i) \\
x_i, p_i & \geq 0
\end{align*}
\]

Note: will need solution to be integral!
The SailCo Problem

SailCo must determine how many sailboats to produce in each quarter in order to meet demand. Boats can be used in same quarter produced, and held-over to future quarter.

Formulate an LP to determine a production schedule for Q1-Q4 to min total production and inventory costs. (Assume there is no demand after Q4.)

<table>
<thead>
<tr>
<th></th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forecast demand</td>
<td>40</td>
<td>60</td>
<td>75</td>
<td>25</td>
</tr>
</tbody>
</table>

**Production cost**

$400/boat, first 40 boats in a quarter

$450/boat for additional boats

**Inventory cost**

$20/boat/quarter for boats on hand at end of a quarter (after production has occurred and demand satisfied)

**Initial inventory:** 10 sailboats at start of Q1

\[
\begin{align*}
\text{min } z &= 400 \sum_t x_t + 450 \sum_t y_t + 20 \sum_t h_t \\
\text{s.t. } x_t &\leq 40, \quad \forall t \\
\quad h_1 &= 10 + x_1 + y_1 - 40 \\
\quad h_2 &= h_1 + x_2 + y_2 - 60 \\
\quad h_3 &= h_2 + x_3 + y_3 - 75 \\
\quad h_4 &= h_3 + x_4 + y_4 - 25 \\
\quad h_t, y_t, x_t &\geq 0
\end{align*}
\]

\(h_t\): represents # boats on hand at end of quarter

\(x_t\): number of boats made up to 40

\(y_t\): number of boats made above 40
Variation 1: A Rolling Horizon

- Suppose make 40 in Q1, and actual demand is 35.
- Start Q2 with 10+40-35=15 boats on hand.
- New planning period is Q2-Q5.
- Currently the optimal solution to the LP will not try to keep boats on hand at end of Q5.
- Modify the formulation to work for Q2-Q5, and ensure that we end the Q5 “planning horizon” with 10 boats in inventory.
- Forecast for Q2-Q5:

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</tr>
</thead>
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<td>40</td>
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<td>75</td>
<td>25</td>
<td>36</td>
</tr>
<tr>
<td>actual demand</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Formulate an LP for Q2-Q5 to determine a production schedule to minimize sum of costs and meet new constraint.

Formulation for Q2-Q5:

\[
\text{min } z = 400 \sum_t x_t + 450 \sum_t y_t + 20 \sum_t h_t \\
\text{s.t. } x_t \leq 40, \quad \forall t \\
\quad h_2 = 15 + x_2 + y_2 - 60 \\
\quad h_3 = h_2 + x_3 + y_3 - 75 \\
\quad h_4 = h_3 + x_4 + y_4 - 25 \\
\quad h_5 = h_4 + x_5 + y_5 - 36 \\
\quad h_5 \geq 10 \\
\quad h_t, y_t, x_t \geq 0
\]

- \( h_i \): represents # boats on hand at end of quarter
- \( x_i \): number of boats made up to 40
- \( y_i \): number of boats made above 40
- Decision variables for periods 2, ..., 5
Variation 2: Production Smoothing

• Make production “smooth” across periods:
  – an increase in production costs $400/boat (training)
  – a decrease in production costs $500/boat (severance pay, loss in morale)
• Assume 50 boats made during the Q preceding Q1, initial inventory of 10.
• Need at least 10 boats on hand at end of planning horizon

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• Formulate an LP for Q1-Q4 to determine a production schedule to minimize sum of production and inventory costs.

\[
\begin{align*}
\min z &= 400 \sum_t x_t + 450 \sum_t y_t + 20 \sum_t h_t + 400 \sum_t c^+_t + 500 \sum_t c^-_t \\
\text{s.t.} & \quad x_t \leq 40, \quad \forall t \\
& \quad h_1 = 10 + x_1 + y_1 - 40 \\
& \quad h_2 = h_1 + x_2 + y_2 - 60 \\
& \quad h_3 = h_2 + x_3 + y_3 - 75 \\
& \quad h_4 = h_3 + x_4 + y_4 - 25 \\
& \quad h_4 \geq 10 \\
& \quad x_1 + y_1 - 50 = c^+_1 - c^-_1 \\
& \quad x_2 + y_2 - (x_1 + y_1) = c^+_2 - c^-_2 \\
& \quad x_3 + y_3 - (x_2 + y_2) = c^+_3 - c^-_3 \\
& \quad x_4 + y_4 - (x_3 + y_3) = c^+_4 - c^-_4 \\
& \quad h_t, y_t, x_t, c^+_t, c^-_t \geq 0
\end{align*}
\]

Solution: \(x=(40,40,40,40); y=(15,15,15,15)\)

\(h_t\): # boats on hand at end of quarter
\(x_t, y_t\): number of boats made up to (above) 40
\(c^+_t\): #boat increase from last period
\(c^-_t\): #boat decrease from last period
• Why will the optimal solution not have both 
  \( c_t^+ > 0 \) and \( c_t^- > 0 \)?
• Suppose production 60 in period 1 and 70 in 
  period 2. What assignments to \( c_2^+ \) and \( c_2^- \) are feasible?
  \[
  70 - 60 = c_2^+ - c_2^-
  \]
  \[c_2^+, c_2^- \geq 0\]
• What assignments are optimal?
  \[
  \text{min } \ldots + 400c_2^+ + 500c_2^- + \ldots
  \]

Variation 3: Allowing demands to be backlogged

• Suppose demands can be met in future periods.
• Penalty $100/boat per quarter demand is unmet.
• Must meet all demand by end of Q4.
• Build from variation 2 (still face smoothing costs 
  with 50 in previous period and 10 on hand at start 
  of Q1; still want 10 on hand at end of Q4.)

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</tbody>
</table>

Formulate an LP to determine a production schedule to minimize the sum of the production and inventory costs.
The SAVE-IT Company

- Operates a recycling center. Collects four types materials, treats, and amalgamates to make an insulation product of three different grades.
- Formulate an LP to determine the amount of each grade and the mix of materials for each grade, maximizing profit (sales – cost)

Have grant of $30,000/wk for treatment cost (can’t treat more than this)

At least half of each type of material must be treated.

<table>
<thead>
<tr>
<th>Material</th>
<th>Availability (lbs/wk)</th>
<th>Treatment cost ($/lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3000</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2000</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>4000</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>1000</td>
<td>5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Grade</th>
<th>Spec</th>
<th>Amalg. cost ($/lb)</th>
<th>Sales price ($/lb)</th>
</tr>
</thead>
</table>
| A     | 1 <= 30%  
2 >= 40%  
3 <= 50%  
4 =20% | 3 | 8.5 |
| B     | 1 <= 50%  
2 >= 10%  
3: any  
4 = 10% | 2.5 | 7 |
| C     | 1 <= 70%  
2,3,4: any | 2 | 5.5 |
\[
\begin{align*}
\text{max} & \quad 5.5 \sum_j x_{A_j} + 4.5 \sum_j x_{B_j} + 3.5 \sum_j x_{C_j} \\
\text{s.t.} & \quad x_{A1} \leq 0.3 \sum_j x_{A_j} \\
& \quad x_{A2} \geq 0.4 \sum_j x_{A_j} \\
& \quad x_{A3} \leq 0.5 \sum_j x_{A_j} \\
& \quad x_{A4} = 0.2 \sum_j x_{A_j} \\
& \quad x_{B1} \leq 0.5 \sum_j x_{B_j} \\
& \quad x_{B2} \geq 0.1 \sum_j x_{B_j} \\
& \quad x_{B4} = 0.1 \sum_j x_{B_j} \\
& \quad x_{C1} \leq 0.7 \sum_j x_{C_j} \\
& \quad 1500 \leq x_{A1} + x_{B1} + x_{C1} \leq 3000 \\
& \quad 1000 \leq x_{A2} + x_{B2} + x_{C2} \leq 2000 \\
& \quad 2000 \leq x_{A3} + x_{B3} + x_{C3} \leq 4000 \\
& \quad 500 \leq x_{A4} + x_{B4} + x_{C4} \leq 1000 \\
& \quad 3(x_{A1} + x_{B1} + x_{C1}) + 6(x_{A2} + x_{B2} + x_{C2}) \\
& + 4(x_{A3} + x_{B3} + x_{C3}) + 5(x_{A4} + x_{B4} + x_{C4}) \leq 30000 \\
& \quad x_{A1}, x_{A2}, \ldots, x_{C4} \geq 0
\end{align*}
\]

**Variation 1: Pay for Treatment**

- SAVE-IT recognizes that it might be able to make more profits and improve the environment.
- Gets matching funds: provide a 80% rebate on treatment cost above $30,000 per week.
- Formulate an LP that (a) uses all of $30,000, (b) treats at least half of each type of material, (c) may treat more material if this is profitable.
• Introduce new constraint:

\[ y = 3(x_{A1} + x_{B1} + x_{C1}) + 6(x_{A2} + x_{B2} + x_{C2}) + \\
6(x_{A3} + x_{B3} + x_{C3}) + 5(x_{A4} + x_{B4} + x_{C4}) - 30000 \]

• Decision variable (total dollar cost of excess treatment) \( y \geq 0 \)

• Add \(-0.2y\) to the objective

• Drop inequality (*)

• The new formulation is on the next slide
A Simple AMPL Example

max   5x_1 + 8x_2 

s.t.   x_1 + x_2 ≤ 6 

5x_1 + 9x_2 ≤ 45 

x_1, x_2 ≥ 0

Product A | Product B
---|---
Profit Per Unit | 5 | 8
Machine time | 1 | 1
Storage Space | 5 | 9

Total machine time = 6; Total storage space = 45;

**example.mod**

```ampl
set PRODUCT := A, B;
set RESOURCE := machine, space;
param usage {i in RESOURCE, j in PRODUCT} :=
    machine  1 1
    space    5 9;
param profit {j in PRODUCT} :=
    A 5  B 8;
param avail {i in RESOURCE} :=
    machine 6  space 45;
var X {j in PRODUCT} >= 0;
maximize Total_Profit : sum {j in PRODUCT} profit[j] * X[j];
subject to Resource_Constraints {i in RESOURCE}:
    sum {j in PRODUCT} usage[i, j] * X[j] <= avail[i];
```

**example.dat**

```dat
set PRODUCT := A, B;
set RESOURCE := machine, space;
param usage :=
    machine 1 1
    space  5 9;
param profit :=
    A 5  B 8;
param avail :=
    machine  6  space  45;
```

Formulate in AMPL, solve with CPLEX

- Install AMPL
- Start AMPL using AMPL IDE or at command line
  ```bash
  ampl: model example.mod;
  ampl: data example.dat;
  ampl: option solver cplex;
  ampl: solve;
  CPLEX 12.5.0.0: optimal solution; objective 41.25
  2 dual simplex iterations (1 in phase I)
  ampl: display X;
  X [*] :=
      A  2.25
      B  3.75
  ;
  ```