AM 121: Intro to Optimization Models and Methods

Fall 2019

Lecture 2: Intro to LP, Linear algebra review.

Yiling Chen
SEAS

Logistics

- Complete the section time survey today
  https://forms.gle/NPtYQUHv9FN6ndBo6
- Complete assignment 0 today
- Office hours will be posted on course website today (always check the page before you go)
- Assignment 1 will be posted today (due Tue. 9/17)

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Lecture 2: Lesson Plan

- What is an LP?
- Graphical and algebraic correspondence
- Problems in canonical form
- LP in matrix form. Matrix review.

Jensen & Bard: 2.1-2.3, 2.5, 3.1 (can ignore the two definitions for now), 3.2

Available in Cabot Science Library.

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Linear Programming

- Maximizing (or minimizing) a linear function subject to a finite number of linear constraints

$$\max \sum_{j=1}^{n} c_j x_j$$

objective function

$$\text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, \ldots, m)$$

constraints

$$x_j \geq 0 \quad (j = 1, \ldots, n)$$

non-negativity

Decision variables: x_j
Parameters: c_j, a_{ij}

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Standard Inequality Form

$$\begin{aligned} \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}x \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

$$\boldsymbol{c}^T = (c_1, \ldots, c_n)$$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

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Standard Equality Form

$$\begin{aligned} \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}x = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

$$\boldsymbol{c}^T = (c_1, \ldots, c_n)$$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

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A Little History

- The field of linear programming started in 1947 when George Dantzig designed the "simplex method" for solving U.S. Air Force planning problems
- Dantzig was deciding how to use the limited resources of the Air Force
- "planning" == "programming"
  - "program" was a military term that referred to plans or proposed schedules for training, logistical supply, or deployment of combat units.
  - this naming sometimes called "Dantzig's great mistake"

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Terminology for Solutions of LP

- A feasible solution
  - A solution that satisfies all constraints
- An infeasible solution
  - A solution that violates at least one constraint
- Feasible region
  - The region of all feasible solutions
- An optimal solution
  - A feasible solution that has the most favorable value of the objective function

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Example: Marketing Campaign

- Ad on news page– get 7m high-income women, 2m high-income men. $50,000
- Ad on sports page– get 2m high-income women and 12m high-income men. $100,000
- Goal: 28m women, 24m men; min cost. How many of each ad to buy? (*Can buy fractions*!)

$$\begin{aligned} \min \quad z &= 50x_1 + 100x_2 \\ \text{s.t.} \quad 7x_1 + 2x_2 &\geq 28 \\ 2x_1 + 12x_2 &\geq 24 \\ x_1, x_2 &\geq 0 \end{aligned}$$

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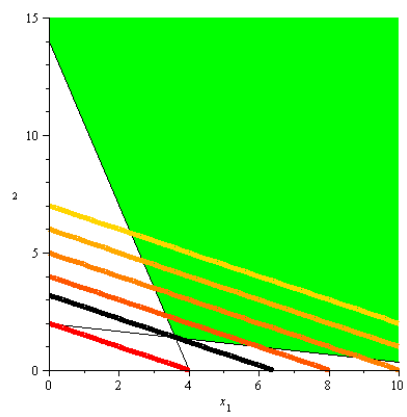
Graphical version of problem
(solution is x_1=3.6, x_2=1.4, value 320)

$$\begin{aligned} \min \quad z &= 50x_1 + 100x_2 \\ \text{s.t.} \quad 7x_1 + 2x_2 &\geq 28 \\ 2x_1 + 12x_2 &\geq 24 \\ x_1, x_2 &\geq 0 \end{aligned}$$



Solution is at an extreme point of feasible region!

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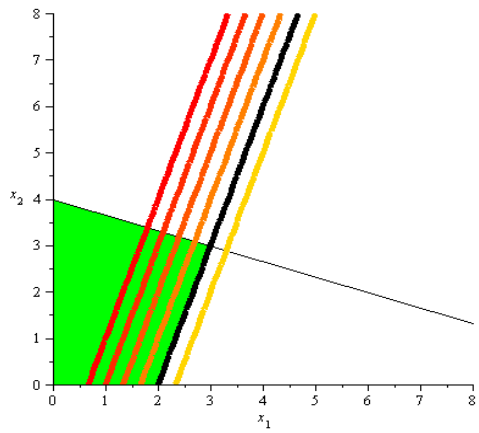
Example: Multiple Opt. Solutions

$$\begin{aligned} \max \quad & 3x_1 - x_2 \\ \text{s.t.} \quad & 15x_1 - 5x_2 \leq 30 \\ & 10x_1 + 30x_2 \leq 120 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Note: still extremal optimal solutions

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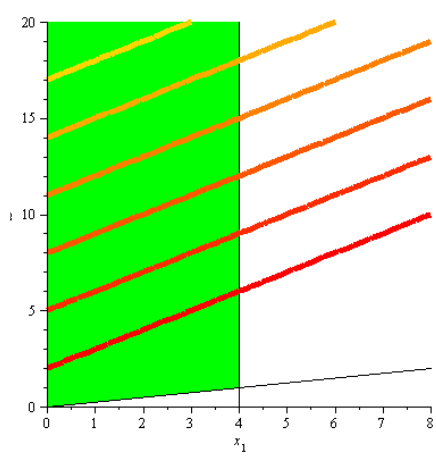
Example: Unbounded Objective

$$\begin{aligned} \max \quad & -x_1 + x_2 \\ \text{s.t.} \quad & -x_1 + 4x_2 \geq 0 \\ & x_1 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$



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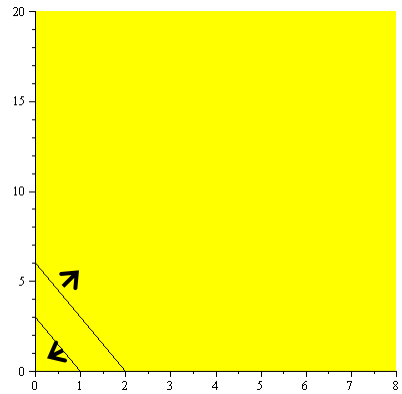
Example: Infeasible Problem

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \geq 6 \\ & 3x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$



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Solving LPs

- Transform to the **canonical form** (note: this is NOT the "standard equality form")
- Work with **basic feasible solutions**
- **Iterate**: solution improvement
  - From one BFS to the next…

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Canonical Form

$$
\begin{array}{rrrrrl}
\max z = & 0x_1 & +0x_2 & -3x_3 & -x_4 & +20 \\
\text{s.t.} & & & & & \\
 & x_1 & & -3x_3 & +3x_4 & = 6 \\
 & & x_2 & -8x_3 & +4x_4 & = 4 \\
 & & x_j \geq 0 & & &
\end{array}
$$

1. **Maximization**
2. **RHS coefficients are non-negative**
3. All constraints are **equalities**
4. **Decision variables all non-negative**
5. One decision variable is "**isolated**" in each constraint:
   - a +1 coefficient.
   - does not appear in any other constraint
   - zero coefficient in objective

Why might this be useful??

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Basic Feasible Solution

$$
\begin{array}{rrrrrl}
\max z = & 0x_1 & +0x_2 & -3x_3 & -x_4 & +20 \\
\text{s.t.} & & & & & \\
 & x_1 & & -3x_3 & +3x_4 & = 6 \\
 & & x_2 & -8x_3 & +4x_4 & = 4 \\
 & & x_j \geq 0 & & &
\end{array}
$$

Canonical form has an associated basic feasible solution in which the isolated variables (basic vars) are <u>non-zero</u> and the rest (non-basic vars) are <u>zero</u>.

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Basic Feasible Solution

$$
\begin{array}{rlllll}
\max z = & 0x_1 & +0x_2 & -3x_3 & -x_4 & +20 \\
\text{s.t.} & & & & & \\
 & x_1 & & -3x_3 & +3x_4 & = 6 \\
 & & x_2 & -8x_3 & +4x_4 & = 4 \\
 & & x_j \geq 0 & & &
\end{array}
$$

Canonical form has an associated basic feasible solution in which the isolated variables (basic vars) are non-zero and the rest (non-basic vars) are zero.

Here, set $x_1 = 6$, $x_2=4$, $x_3=0$, $x_4=0$.

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Basic Feasible Solution

$$
\begin{array}{rlllll}
\max z = & 0x_1 & +0x_2 & -3x_3 & -x_4 & +20 \\
\text{s.t.} & & & & & \\
 & x_1 & & -3x_3 & +3x_4 & = 6 \\
 & & x_2 & -8x_3 & +4x_4 & = 4 \\
 & & x_j \geq 0 & & &
\end{array}
$$

Canonical form has an associated basic feasible solution in which the isolated variables (basic vars) are non-zero and the rest (non-basic vars) are zero.

Here, set $x_1 = 6$, $x_2=4$, $x_3=0$, $x_4=0$.

Optimal in this example as well. (Why?)

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Solution Improvement

$$
\begin{array}{lrrrrl}
\max z = & 0x_1 & +0x_2 & -3x_3 & \boxed{+x_4} & +20 \\
\text{s.t.} & & & & & \\
 & x_1 & & -3x_3 & +3x_4 & = 6 \\
 & & x_2 & -8x_3 & +4x_4 & = 4 \\
 & & x_j \geq 0 & & &
\end{array}
$$

Current BFS: $x_1 = 6$, $x_2=4$, $x_3=0$, $x_4=0$.

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Solution Improvement

$$
\begin{array}{lrrrrl}
\max z = & 0x_1 & +0x_2 & -3x_3 & \boxed{+x_4} & +20 \\
\text{s.t.} & & & & & \\
 & x_1 & & -3x_3 & +3x_4 & = 6 \\
 & & x_2 & -8x_3 & +4x_4 & = 4 \\
 & & x_j \geq 0 & & &
\end{array}
$$

Current BFS: $x_1 = 6$, $x_2=4$, $x_3=0$, $x_4=0$.
Let's increase x_4. Need to decrease x_1 and x_2 (keep $x_3=0$) to keep feasible.

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Solution Improvement

$$
\begin{array}{lrrrrl}
\max z = & 0x_1 & +0x_2 & -3x_3 & \boxed{+x_4} & +20 \\
\text{s.t.} & & & & & \\
 & x_1 & & -3x_3 & +3x_4 & = 6 \\
 & & x_2 & -8x_3 & +4x_4 & = 4 \\
 & & x_j \geq 0 & & &
\end{array}
$$

Current BFS: $x_1 = 6$, $x_2=4$, $x_3=0$, $x_4=0$.
Let's increase x_4. Need to decrease x_1 and x_2 (keep $x_3=0$) to keep feasible. Second constraint becomes <u>binding</u>.
Obtain new solution: $x_1=3$, $x_2=0$, $x_3=0$, $x_4=1$. Value 21.

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Solution Improvement

$$
\begin{array}{lrrrrl}
\max z = & 0x_1 & +0x_2 & -3x_3 & \boxed{+x_4} & +20 \\
\text{s.t.} & & & & & \\
 & x_1 & & -3x_3 & +3x_4 & = 6 \\
 & & x_2 & -8x_3 & +4x_4 & = 4 \\
 & & x_j \geq 0 & & &
\end{array}
$$

Current BFS: $x_1 = 6$, $x_2=4$, $x_3=0$, $x_4=0$.
Let's increase x_4. Need to decrease x_1 and x_2 (keep $x_3=0$) to keep feasible. Second constraint becomes <u>binding</u>.
Obtain new solution: $x_1=3$, $x_2=0$, $x_3=0$, $x_4=1$. Value 21.

Corresponds to a new <u>canonical form</u>. Isolated vars: x_1 and x_4.

"*pivot on x_4 in the second constraint*"

"*pick something to enter, something forced to leave*" 22

New Canonical Form

After linear transformations:

$$
\begin{array}{lllllll}
\max z = & 0x_1 & -\frac{1}{4}x_2 & -x_3 & +0x_4 & +21 \\
\text{s.t.} & & & & & \\
 & x_1 & -\frac{3}{4}x_2 & +3x_3 & & = 3 \\
 & & \frac{1}{4}x_2 & -2x_3 & +x_4 & = 1 \\
 & & x_j \geq 0 & & &
\end{array}
$$

New BFS is x_1=3, x_2=0, x_3=0, x_4=1, and optimal.

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Geometric Interpretation of Solution Improvement

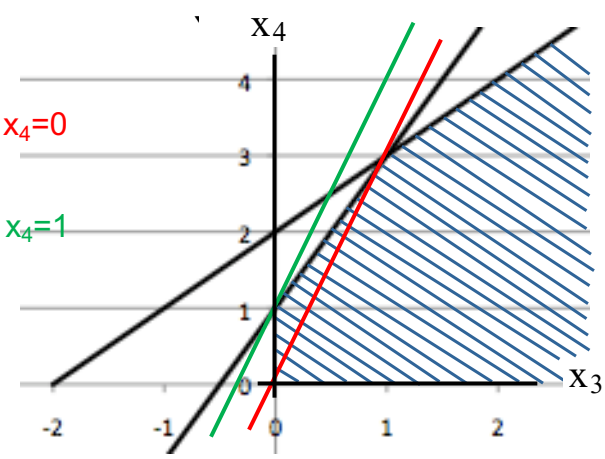
$$
\begin{array}{lllllll}
\max z = & 0x_1 & +0x_2 & -3x_3 & +x_4 & +20 \\
\text{s.t.} & & & & & \\
 & x_1 & & -3x_3 & +3x_4 & = 6 \\
 & & x_2 & -8x_3 & +4x_4 & = 4 \\
 & & x_j \geq 0 & & &
\end{array}
$$

$$x_1 = 3x_3 - 3x_4 + 6 \geq 0$$
$$x_2 = 8x_3 - 4x_4 + 4 \geq 0$$

x_1=6, x_2=4, x_3=0, x_4=0

⇩

x_1=3, x_2=0, x_3=0, x_4=1



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Can any LP be made canonical?

$$
\begin{array}{lllllll}
\max z = & 0x_1 & -\frac{1}{4}x_2 & -x_3 & +0x_4 & +21 \\
\text{s.t.} & & & & & \\
 & x_1 & -\frac{3}{4}x_2 & +3x_3 & & = 3 \\
 & & \frac{1}{4}x_2 & -2x_3 & +x_4 & = 1 \\
 & & x_j \geq 0 & & &
\end{array}
$$

(1) maximization, (2) positive RHS, (3) equality constraints, (4) non-negative vars, (5) isolated vars.

+1 coeff, only in one constraint, not in obj.

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Reduction to canonical form (I)

- "min z" =
- If a RHS value is negative then
- If x_1 <= 0 then
- If x_3 is "free" (neither x_3 <= 0 or $x_3 \geq 0$) then

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Reduction to canonical form (I)

- “min z” = “max –z”
- If a RHS value is negative then

- If x_1 <= 0 then

- If x_3 is “free” (neither x_3 <= 0 or $x_3 \geq 0$) then

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Reduction to canonical form (I)

- “min z” = “max –z”
- If a RHS value is negative then multiply constraint by -1

- If x_1 <= 0 then

- If x_3 is “free” (neither x_3 <= 0 or $x_3 \geq 0$) then

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Reduction to canonical form (I)

- “min z” = “max –z”
- If a RHS value is negative then multiply constraint by -1
- If $x_1 <= 0$ then replace $x_1 := -x_2$, with $x_2 \geq 0$
- If x_3 is “free” (neither $x_3 <= 0$ or $x_3 \geq 0$) then

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Reduction to canonical form (I)

- “min z” = “max –z”
- If a RHS value is negative then multiply constraint by -1
- If $x_1 <= 0$ then replace $x_1 := -x_2$, with $x_2 \geq 0$
- If x_3 is “free” (neither $x_3 <= 0$ or $x_3 \geq 0$) then replace $x_3 := u - v$, with $u \geq 0$ and $v \geq 0$.

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Reduction to canonical form (II)

- Inequality constraints

$$40x_1 + 10x_2 + 6x_3 \leq 55$$
$$40x_1 + 10x_2 + 6x_3 \geq 33$$

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Reduction to canonical form (II)

- Inequality constraints

$$40x_1 + 10x_2 + 6x_3 \leq 55$$
$$40x_1 + 10x_2 + 6x_3 \geq 33$$

$$\begin{array}{lllllll} 40x_1 & +10x_2 & +6x_3 & +x_4 & & = 55 \\ 40x_1 & +10x_2 & +6x_3 & & -x_5 & = 33 \end{array}$$

slack variable surplus variable

$x_4 \geq 0$, $x_5 \geq 0$

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Reduction to canonical form (III)

- Need isolated variables
- A constraint with slack var already good!

$$40x_1 \quad +10x_2 \quad +6x_3 \quad +x_4 \qquad\qquad = 55$$

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Reduction to canonical form (III)

- Need isolated variables
- A constraint with slack var already good!

$$40x_1 \quad +10x_2 \quad +6x_3 \quad +x_4 \qquad\qquad = 55$$

- Other constraints, e.g. with surplus vars not good:

$$40x_1 \quad +10x_2 \quad +6x_3 \qquad\qquad -x_5 \quad = 33$$

doesn't work

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Reduction to canonical form (III)

- Need isolated variables
- A constraint with slack var already good!

$$40x_1 \quad +10x_2 \quad +6x_3 \quad +x_4 \qquad\qquad = 55$$

- Other constraints, e.g. with surplus vars not good:

$$40x_1 \quad +10x_2 \quad +6x_3 \qquad\qquad -x_5 \quad = 33$$

doesn't work

- Introduce a new <u>artificial variable</u> (we'll insist that x_6=0 in any solution)

$$40x_1 \quad +10x_2 \quad +6x_3 \qquad -x_5 \quad +x_6 \quad = 33$$

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Standard Inequality Form

$$\begin{aligned} \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}x \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

$$\boldsymbol{c}^T = (c_1, \ldots, c_n)$$

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

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Review: Matrices (1/4)

- Matrix: rectangular array of numbers $[a_{ij}]$
  - dimension: *m by n* (*m* rows, *n* columns)
  - *k by* 1: column vector; 1 *by k*: row vector
- $\boldsymbol{B} = \alpha\boldsymbol{A} = \boldsymbol{A}\alpha$, scalar α: $\alpha a_{ij} = b_{ij}$

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Review: Matrices (1/4)

- Matrix: rectangular array of numbers $[a_{ij}]$
  - dimension: *m by n* (*m* rows, *n* columns)
  - *k by* 1: column vector; 1 *by k*: row vector
- $\boldsymbol{B} = \alpha\boldsymbol{A} = \boldsymbol{A}\alpha$, scalar α: $\alpha a_{ij} = b_{ij}$

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{C}$$

(m x p) (p x n) (m x n)

$$\begin{pmatrix} 2 & 6 & -3 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -3 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} -7 & -17 \\ 1 & -10 \end{pmatrix}$$

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Review: Matrices (2/4)

- $\boldsymbol{A}^T$ transpose: $a^T_{ij} = a_{ji}$

$$A = \begin{pmatrix} 2 & 4 & -1 \\ -3 & 0 & 44 \end{pmatrix} \quad A^T = \begin{pmatrix} 2 & -3 \\ 4 & 0 \\ -1 & 4 \end{pmatrix}$$

- $\boldsymbol{c}^T \cdot x = \sum_{j=1}^{n} c_j x_j$ "inner product"

(1 x n) (n x 1)

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Review: Matrices (2/4)

- $\boldsymbol{A}^T$ transpose: $a^T_{ij} = a_{ji}$

$$A = \begin{pmatrix} 2 & 4 & -1 \\ -3 & 0 & 44 \end{pmatrix} \quad A^T = \begin{pmatrix} 2 & -3 \\ 4 & 0 \\ -1 & 4 \end{pmatrix}$$

- $\boldsymbol{c}^T \cdot x = \sum_{j=1}^{n} c_j x_j$ "inner product"

(1 x n) (n x 1)

- Partitions

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \left(\begin{array}{c|c|c} A_1 & \cdots & A_n \end{array} \right)$$

$$Ax = A_1 x_1 + \ldots + A_n x_n$$

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Review: Matrices (3/4)

- **Square** matrix: *m by m*
- **Identity** matrix: square matrix w/ diagonal elements all 1 and all non-diagonal are 0.
- $\boldsymbol{I}_2$, $\boldsymbol{I}_3$, …

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- *m* by *m* square $\boldsymbol{A}$, **inverse**: $\boldsymbol{A}^{-1} = \boldsymbol{B} \rightarrow \boldsymbol{BA} = \boldsymbol{AB} = \boldsymbol{I}_m$

$$\begin{pmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -5 & 1 & -7 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Review: Matrices (4/4)

- Given $\boldsymbol{Ax} = \boldsymbol{b}$ (with square matrix $\boldsymbol{A}$)
- Can write:

  $\boldsymbol{A}^{-1}(\boldsymbol{Ax})=\boldsymbol{A}^{-1}\boldsymbol{b}$

  Equivalently:

  $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$

- Can find a unique solution to a square linear system if $\boldsymbol{A}$ is invertible.

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Next Time

- Applications, Examples, Exercises.