AM 121: Intro to Optimization Models and Methods

Fall 2018

Lecture 2: Intro to LP, Linear algebra review.

Yiling Chen
SEAS

Lecture 2: Lesson Plan

• What is an LP?
• Graphical and algebraic correspondence
• Problems in canonical form
• LP in matrix form. Matrix review.

Jensen & Bard: 2.1-2.3, 2.5, 3.1 (can ignore the two definitions for now), 3.2

Available in Cabot Science Library.
Linear Programming

- Maximizing (or minimizing) a linear function subject to a finite number of linear constraints

\[
\begin{align*}
\text{max} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, \ldots, m) \\
& x_j \geq 0 \quad (j = 1, \ldots, n)
\end{align*}
\]

Decision variables: \(x_j\)
Parameters: \(c_j, a_{ij}\)

Standard Inequality Form

\[
\begin{align*}
\text{max} \quad & \mathbf{c}^T \mathbf{x} \\
\text{s.t.} \quad & \mathbf{A} \mathbf{x} \leq \mathbf{b} \\
& \mathbf{x} \geq 0 \\
\mathbf{c}^T = (c_1, \ldots, c_n) \\
\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \\
\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\
\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}
\end{align*}
\]
Standard Equality Form

\[
\begin{align*}
\text{max} & \quad \mathbf{c}^T \mathbf{x} \\
\text{s.t.} & \quad A \mathbf{x} = \mathbf{b} \\
& \quad \mathbf{x} \geq 0
\end{align*}
\]

\[
\begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{m1} & \cdots & a_{mn}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix}
=
\begin{pmatrix}
b_1 \\
\vdots \\
b_m
\end{pmatrix}
\]

A Little History

• The field of linear programming started in 1947 when George Dantzig designed the “simplex method” for solving U.S. Air Force planning problems.

• Dantzig was deciding how to use the limited resources of the Air Force.

• “planning” == “programming”
  – “program” was a military term that referred to plans or proposed schedules for training, logistical supply, or deployment of combat units.
  – this naming sometimes called “Dantzig’s great mistake”
Terminology for Solutions of LP

• A feasible solution
  – A solution that satisfies all constraints

• An infeasible solution
  – A solution that violates at least one constraint

• Feasible region
  – The region of all feasible solutions

• An optimal solution
  – A feasible solution that has the most favorable value of the objective function

Example: Marketing Campaign

• Ad on news page– get 7m high-income women, 2m high-income men. $50,000

• Ad on sports page– get 2m high-income women and 12m high-income men. $100,000

• Goal: 28m women, 24m men; min cost. How many of each ad to buy? (Can buy fractions!)

\[
\begin{align*}
\text{min } & \quad z = 50x_1 + 100x_2 \\
\text{s.t. } & \quad 7x_1 + 2x_2 \geq 28 \\
& \quad 2x_1 + 12x_2 \geq 24 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]
Graphical version of problem
(solution is $x_1=3.6$, $x_2=1.4$, value 320)

$$\begin{align*}
\text{min} \quad z &= 50x_1 + 100x_2 \\
\text{s.t.} \quad 7x_1 + 2x_2 &\geq 28 \\
&\quad 2x_1 + 12x_2 &\geq 24 \\
&\quad x_1, x_2 &\geq 0
\end{align*}$$

Solution is at an extreme point of feasible region!

Example: Multiple Opt. Solutions

$$\begin{align*}
\text{max} \quad 3x_1 - x_2 \\
\text{s.t.} \quad 15x_1 - 5x_2 &\leq 30 \\
&\quad 10x_1 + 30x_2 &\leq 120 \\
&\quad x_1, x_2 &\geq 0
\end{align*}$$

Note: still extremal optimal solutions
Example: Unbounded Objective

\[
\begin{align*}
\text{max} & \quad -x_1 + x_2 \\
\text{s.t.} & \quad -x_1 + 4x_2 \geq 0 \\
& \quad x_1 \leq 4 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Example: Infeasible Problem

\[
\begin{align*}
\text{max} & \quad x_1 + x_2 \\
\text{s.t.} & \quad 3x_1 + x_2 \geq 6 \\
& \quad 3x_1 + x_2 \leq 3 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]
Solving LPs

- Transform to the **canonical form** (note: this is NOT the “standard equality form”)
- Work with **basic feasible solutions**
- **Iterate**: solution improvement
  - From one BFS to the next…

---

**Canonical Form**

\[
\begin{align*}
\max z &= 0x_1 + 0x_2 - 3x_3 - x_4 + 20 \\
\text{s.t.} & \\
& \quad x_1 - 3x_3 + 3x_4 = 6 \\
& \quad x_2 - 8x_3 + 4x_4 = 4 \\
& \quad x_j \geq 0
\end{align*}
\]

1. **Maximization**
2. **RHS coefficients are non-negative**
3. All constraints are **equalities**
4. **Decision variables all non-negative**
5. One decision variable is “**isolated**” in each constraint:
   - a +1 coefficient.
   - does not appear in any other constraint
   - zero coefficient in objective

**Why might this be useful??**
Canonical form has an associated basic feasible solution in which the isolated variables (basic vars) are non-zero and the rest (non-basic vars) are zero.

Here, set $x_1 = 6$, $x_2 = 4$, $x_3 = 0$, $x_4 = 0$. 
Basic Feasible Solution

\[ \text{max } z = 0x_1 + 0x_2 - 3x_3 - x_4 + 20 \]

\[ \text{s.t.} \]
\[ x_1 \]
\[ x_2 \]
\[ x_3 \geq 0 \]
\[ -3x_3 + 3x_4 = 6 \]
\[ -8x_3 + 4x_4 = 4 \]

Canonical form has an associated basic feasible solution in which the isolated variables (basic vars) are non-zero and the rest (non-basic vars) are zero.

Here, set \( x_1 = 6, x_2=4, x_3=0, x_4=0 \).

Optimal in this example as well. (Why?)

Solution Improvement

\[ \text{max } z = 0x_1 + 0x_2 - 3x_3 + x_4 + 20 \]

\[ \text{s.t.} \]
\[ x_1 \]
\[ x_2 \]
\[ x_3 \geq 0 \]
\[ -3x_3 + 3x_4 = 6 \]
\[ -8x_3 + 4x_4 = 4 \]

Current BFS: \( x_1 = 6, x_2=4, x_3=0, x_4=0 \).
Solution Improvement

\[ \text{max } z = 0x_1 + 0x_2 - 3x_3 + x_4 + 20 \]
\[ \text{s.t.} \]
\[ x_1 \]
\[ x_2 \]
\[ x_3 \]
\[ x_4 \geq 0 \]

Current BFS: \( x_1 = 6, x_2 = 4, x_3 = 0, x_4 = 0 \).

Let’s increase \( x_4 \). Need to decrease \( x_1 \) and \( x_2 \) (keep \( x_3 = 0 \)) to keep feasible.

Obtain new solution: \( x_1 = 3, x_2 = 0, x_3 = 0, x_4 = 1 \). Value 21.
Solution Improvement

\[
\begin{align*}
\text{max } z &= 0x_1 + 0x_2 - 3x_3 + x_4 + 20 \\
\text{s.t.} & \\
& x_1 - 3x_3 + 3x_4 = 6 \\
& x_2 - 8x_3 + 4x_4 = 4 \\
& x_j \geq 0
\end{align*}
\]

Current BFS: \(x_1 = 6, x_2 = 4, x_3 = 0, x_4 = 0\).
Let’s increase \(x_4\). Need to decrease \(x_1\) and \(x_2\) (keep \(x_3 = 0\)) to keep feasible. Second constraint becomes binding.
Obtain new solution: \(x_1 = 3, x_2 = 0, x_3 = 0, x_4 = 1\). Value 21.

Corresponds to a new canonical form. Isolated vars: \(x_1\) and \(x_4\).
“pivot on \(x_4\) in the second constraint”
“pick something to enter, something forced to leave”

New Canonical Form

After linear transformations:

\[
\begin{align*}
\text{max } z &= 0x_1 - \frac{1}{4}x_2 - x_3 + 0x_4 + 21 \\
\text{s.t.} & \\
& x_1 - \frac{3}{4}x_2 + 3x_3 = 3 \\
& \frac{1}{4}x_2 - 2x_3 + x_4 = 1 \\
& x_j \geq 0
\end{align*}
\]

New BFS is \(x_1 = 3, x_2 = 0, x_3 = 0, x_4 = 1\), and optimal.
Geometric Interpretation of Solution Improvement

\[
\begin{align*}
\max z &= 0x_1 + 0x_2 - 3x_3 + x_4 + 20 \\
\text{s.t.} \quad x_1 &= -3x_3 + 3x_4 = 6 \\
\quad x_2 &= -8x_3 + 4x_4 = 4 \\
\quad x_j &\geq 0
\end{align*}
\]

\[
\begin{align*}
 x_1 &= 3x_3 - 3x_4 + 6 \geq 0 \\
 x_2 &= 8x_3 - 4x_4 + 4 \geq 0
\end{align*}
\]

\[x_1 = 6, \ x_2 = 4, \ x_3 = 0, \ x_4 = 0\]
\[\downarrow\]
\[x_1 = 3, \ x_2 = 0, \ x_3 = 0, \ x_4 = 1\]

Can any LP be made canonical?

\[
\begin{align*}
\max z &= 0x_1 - \frac{1}{4}x_2 - x_3 + 0x_4 + 21 \\
\text{s.t.} \quad x_1 &= -\frac{3}{4}x_2 + 3x_3 = 3 \\
\quad \frac{1}{4}x_2 &= -2x_3 + x_4 = 1 \\
\quad x_j &\geq 0
\end{align*}
\]

\(1\) maximization, \(2\) positive RHS, \(3\) equality constraints, \(4\) non-negative vars, \(5\) isolated vars.

\(+1\) coeff, only in one constraint, not in obj.
Reduction to canonical form (I)

• “min z” =
• If a RHS value is negative then

• If $x_1 \leq 0$ then

• If $x_3$ is “free” (neither $x_3 \leq 0$ or $x_3 \geq 0$) then

Reduction to canonical form (I)

• “min z” = “max –z”
• If a RHS value is negative then

• If $x_1 \leq 0$ then

• If $x_3$ is “free” (neither $x_3 \leq 0$ or $x_3 \geq 0$) then
Reduction to canonical form (I)

• “min z” = “max –z”
• If a RHS value is negative then multiply constraint by -1

• If \( x_1 \leq 0 \) then

• If \( x_3 \) is “free” (neither \( x_3 \leq 0 \) or \( x_3 \geq 0 \)) then replace \( x_3 := u - v \), with \( u \geq 0 \) and \( v \geq 0 \).

Reduction to canonical form (I)

• “min z” = “max –z”
• If a RHS value is negative then multiply constraint by -1

• If \( x_1 \leq 0 \) then replace \( x_1 := -x_2 \), with \( x_2 \geq 0 \)

• If \( x_3 \) is “free” (neither \( x_3 \leq 0 \) or \( x_3 \geq 0 \)) then
Reduction to canonical form (I)

• “min z” = “max –z”
• If a RHS value is negative then multiply constraint by -1
• If \( x_1 \leq 0 \) then replace \( x_1 := -x_2 \), with \( x_2 \geq 0 \)
• If \( x_3 \) is “free” (neither \( x_3 \leq 0 \) or \( x_3 \geq 0 \)) then replace \( x_3 := u – v \), with \( u \geq 0 \) and \( v \geq 0 \).

Reduction to canonical form (II)

• Inequality constraints

\[
\begin{align*}
40x_1 + 10x_2 + 6x_3 & \leq 55 \\
40x_1 + 10x_2 + 6x_3 & \geq 33
\end{align*}
\]
Reduction to canonical form (II)

- Inequality constraints

\[
\begin{align*}
40x_1 + 10x_2 + 6x_3 & \leq 55 \\
40x_1 + 10x_2 + 6x_3 & \geq 33
\end{align*}
\]

\[
\begin{align*}
40x_1 & +10x_2 & +6x_3 & +x_4 & = 55 \\
40x_1 & +10x_2 & +6x_3 & -x_5 & = 33
\end{align*}
\]

\[x_4 \geq 0, \ x_5 \geq 0\]

Reduction to canonical form (III)

- Need isolated variables
- A constraint with slack var already good!

\[
\begin{align*}
40x_1 & +10x_2 & +6x_3 & +x_4 & = 55
\end{align*}
\]
Reduction to canonical form (III)

• Need isolated variables
• A constraint with slack var already good!

\[ 40x_1 + 10x_2 + 6x_3 + x_4 = 55 \]

• Other constraints, e.g. with surplus vars not good:

\[ 40x_1 + 10x_2 + 6x_3 - x_5 = 33 \]

doesn’t work

• Introduce a new artificial variable (we’ll insist that \( x_6 = 0 \) in any solution)

\[ 40x_1 + 10x_2 + 6x_3 - x_5 + x_6 = 33 \]
Standard Inequality Form

\[
\max \quad c^T x \\
\text{s.t.} \quad A x \leq b \\
x \geq 0
\]

\[
c^T = (c_1, \ldots, c_n)
\]

\[
A = \begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{m1} & \cdots & a_{mn}
\end{pmatrix}
\quad x = \begin{pmatrix}
x_1 \\
\vdots \\
x_n
\end{pmatrix}
\quad b = \begin{pmatrix}
b_1 \\
\vdots \\
b_m
\end{pmatrix}
\]

Review: Matrices (1/4)

• Matrix: rectangular array of numbers \([a_{ij}]\)
  – dimension: \(m \times n\) \((m\ \text{rows}, \ n\ \text{columns})\)
  – \(k \times 1\): column vector; \(1 \times k\): row vector

• \(B = \alpha A = A\alpha\), scalar \(\alpha\): \(\alpha a_{ij} = b_{ij}\)
Review: Matrices (1/4)

- Matrix: rectangular array of numbers \([a_{ij}]\)
  - dimension: \(m \times n\) (\(m\) rows, \(n\) columns)
  - \(k \times 1\): column vector; \(1 \times k\): row vector

- \(B = \alpha A = A\alpha\), scalar \(\alpha\): \(\alpha a_{ij} = b_{ij}\)

\[
A \cdot B = C
\]

\[
\begin{pmatrix}
2 & 6 & -3 \\
1 & 4 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
0 & -3 \\
3 & 1
\end{pmatrix}
= \begin{pmatrix}
-7 & -17 \\
1 & -10
\end{pmatrix}
\]

Review: Matrices (2/4)

- \(A^T\) transpose: \(a^T_{ij} = a_{ji}\)

\[
A = \begin{pmatrix}
2 & 4 & -1 \\
-3 & 0 & 44
\end{pmatrix}
A^T = \begin{pmatrix}
2 & -3 \\
4 & 0 \\
-1 & 4
\end{pmatrix}
\]

- \(c^T \cdot x = \sum_{j=1}^{n} c_j x_j\) \((1 \times n) \times(n \times 1)\) \text{“inner product”}
Review: Matrices (2/4)

• \( A^T \) transpose: \( a^T_{ij} = a_{ji} \)

\[
A = \begin{pmatrix}
2 & 4 & -1 \\
-3 & 0 & 4 \\
4 & -1 & 4
\end{pmatrix} \quad A^T = \begin{pmatrix}
2 & -3 \\
4 & 0 \\
-1 & 4
\end{pmatrix}
\]

• \( c^T \cdot x = \sum_{j=1}^{n} c_j x_j \) “inner product”

Partitions

\[
A = \begin{pmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{m1} & \cdots & a_{mn}
\end{pmatrix} = \begin{pmatrix}
A_1 & \cdots & A_n
\end{pmatrix}
\]

\[Ax = A_1 x_1 + \ldots + A_n x_n\]

Review: Matrices (3/4)

• **Square** matrix: \( m \) by \( m \)

• **Identity** matrix: square matrix w/ diagonal elements all 1 and all non-diagonal are 0.

\[
I_2, I_3, \ldots 
I_2 = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \quad I_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

• \( m \) by \( m \) square \( A \), inverse: \( A^{-1} = B \Rightarrow BA = AB = I_m \)

\[
\begin{pmatrix}
2 & 0 & -1 \\
3 & 1 & 2 \\
-1 & 0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 & 1 \\
-5 & 1 & -7 \\
1 & 0 & 2
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
Review: Matrices (4/4)

• Given $Ax = b$ (with square matrix $A$)
• Can write:
  $$A^{-1}(Ax) = A^{-1}b$$
  
  Equivalently:
  $$x = A^{-1}b$$

• Can find a unique solution to a square linear system if $A$ is invertible.

Next Time

• Applications, Examples, Exercises.