Lecture 16: More cuts, Branch and Cut, other tricks…

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Lesson Plan

• Review Chvatal-Gomory cuts
• Cover cuts
• Branch-and-cut
• Other tricks
  – Preprocessing
  – Primal heuristics
  – SOS branching

Jensen & Bard: 8.5
Chvátal-Gomory inequalities

- Consider $X = P \cap \mathbb{Z}^2$, where $P$ is given by
  
  $7x_1 - 2x_2 \leq 14$
  $x_2 \leq 3$
  $2x_1 - 2x_2 \leq 3$
  $x \geq 0$

- Combine with weights $u=(2/7, 37/63, 0)$
  
  $2x_1 + 1/63x_2 \leq 121/21$ (*)

- Round coefficients on LHS down to nearest integer
  
  $2x_1 \leq 5$

Example (Gomory’s algorithm)

- Consider the IP
  
  $z = \max 4x_1 - x_2$
  $7x_1 - 2x_2 \leq 14$
  $x_2 \leq 3$
  $2x_1 - 2x_2 \leq 3$
  $x_1, \quad x_2 \geq 0$, integer
For row i with fractional RHS, the CG cut is
\[ \sum_{j \in B} \pi_j x_j \geq \pi_0; \] with \( \pi_j = \bar{a}_{ij} - \bar{a}_{ij} \) and \( \pi_0 = \bar{b}_i - \bar{b}_i \)

- \( z + 4/7 x_3 + 1/7 x_4 = 59/7 \)
- \( x_1 + 1/7x_3 + 2/7 x_4 = 20/7 \)
- \( x_2 + x_4 = 3 \)
- \( -2/7 x_3 + 10/7x_4 + x_5 = 23/7 \)

- \( B=\{1, 2, 5\} \)
- Add cut \( 1/7x_3 + 2/7x_4 \geq 6/7 \)

- Add cut, re-optimize. New optimal tableau:
  - \( z + \frac{1}{2}x_5 + 3x_6 = 15/2 \)
  - \( x_1 + x_6 = 2 \)
  - \( x_2 - \frac{1}{2}x_5 + x_6 = \frac{1}{2} \)
  - \( x_3 - x_5 - 5x_6 = 1 \)
  - \( x_4 + \frac{1}{2}x_5 + 6x_6 = 5/2 \)

- \( B=\{1,2,3,4\} \) and \( x^*=(2, 1/2, 1, 5/2, 0, 0) \)
- Add cut \( \frac{1}{2}x_5 \geq \frac{1}{2} \)
• Add cut, re-optimize. New optimal tableau:

\[
\begin{align*}
   z & = 3x_6 + x_7 = 7 \\
   x_1 & = x_6 = 2 \\
   x_2 & = x_6 - x_7 = 1 \\
   x_3 & = -5x_6 - 2x_7 = 2 \\
   x_4 & = 6x_6 + x_7 = 2 \\
   x_5 & = -x_7 = 1
\end{align*}
\]

• \( x^* = (2, 1, 2, 2, 1, 0, 0) \)
Cover inequalities

- Defined for problems in which the feasible space is a “0-1 Knapsack set”. Example:
  \[ X = \{ x \in \{0,1\}^7 : 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \} \]

  - Some (minimal) cover inequalities for \( X \) are:
    \[
    \begin{align*}
    x_1 + x_2 + x_3 & \leq 2 \\
    x_1 + x_2 & + x_6 \leq 2 \\
    x_1 & + x_5 + x_6 \leq 2 \\
    x_3 + x_4 + x_5 + x_6 & \leq 3
    \end{align*}
    \]

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Cover inequalities

- For \( X = \{ x \in \{0,1\}^n : \sum_{j} a_j x_j \leq b \} \), with \( a_j > 0, \ b > 0 \).
- \( N = \{1,\ldots,n\} \)
- **Defn.** A set \( C \subseteq N \) is a cover if \( \sum_{j \in C} a_j > b \). A cover is minimal if \( C \setminus \{j\} \) is not a cover for any \( j \in C \).
- **Proposition.** If \( C \) is a cover, the cover inequality \( \sum_{j \in C} x_j \leq |C|-1 \) is valid.
- **Proof.** Consider \( x' \in \{0,1\}^n \) with \( \sum_{j \in C} x'_j > |C|-1 \). Have \( \sum_{j} a_j x'_j \geq \sum_{j \in C} a_j x'_j = \sum_{j \in C} a_j > b \)
  \[ a_j > 0 \quad \text{integrality} \quad \text{cover} \]

  Therefore, this \( x' \) is infeasible.
Two Variations on Cover Inequalities

- Extended Cover Inequalities
- Lifted Cover Inequalities

Extended Cover Inequalities

- \( X=\{x \in \{0,1\}^7 : 11x_1+6x_2+6x_3+5x_4+5x_5+4x_6+x_7 \leq 19 \} \)
- Cover inequality: \( x_3 + x_4 + x_5 + x_6 \leq 3 \)
- Extended cover ineq: \( x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3 \)
- Still valid. Why? Any four variables have sum coefficients \( > 19 \).

**Rule**: Can extend \( C \) to \( E(C) \), where

\[ E(C) = \{ j : a_j \geq a_i \text{ for all } i \in C \} \]

- Still valid; stronger.
Lifted Cover Inequalities

• $X = \{ x \in \{0,1\}^7 : 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19 \}$

• **Lifting:** Given $x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$, find a valid inequality $\alpha_1 x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$, for some $\alpha_1 > 0$.

• Valid for any soln with $x_1 = 0$. If $x_1 = 1$, we know $x$ satisfies: $6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 8$ (*)

• **Idea:** pick maximal $\alpha_1$ s.t. $\alpha_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$ for all integer solutions satisfying (*)

• Set $\alpha_1 = 3 - \max_x \{ x_2 + x_3 + x_4 + x_5 + x_6 : 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 8, x \in \{0,1\}^5 \} = 3 - 1 = 2$

• **Lifted cover inequality:** $2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$

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• **Theorem.** Lifted cover inequalities are “facet-defining” for $\text{conv}(X)$.

• (Informal) An inequality is **facet defining** if it defines the face of a polyhedron and is as high dimension as possible. Consider:
How can we use Cover inequalities?

• **Defn.** A cut is a valid inequality that separates the current fractional solution $x^*$.
Separation with Cover inequalities

- Can write cover inequality as
  \[ \sum_{j \in C} (1 - x_j) \geq 1 \] ("at least one not used")
- Given \( x^* \), find valid \( C \) with \( \sum_{j \in C} (1-x_j^*) < 1 \)

- Formulate an IP. Let \( z_j \) denote whether \( j \in C \).
  \[
  \beta = \min \sum_{j \in N} (1-x_j^*)z_j \\
  \text{s.t.} \quad \sum_{j \in N} a_j z_j > b \quad \text{(valid cover)}
  \]
  \( z \in \{0,1\}^n \)
- If \( \beta \geq 1 \) then \( x^* \) satisfies all cover inequalities, if \( \beta < 1 \) then we find a cut.

  Note: this IP is nonstandard (strict inequality.) But can replace with \( \sum_{j \in N} a_j z_j \geq b+1 \) when coeffs are integral.

Example: Separation

- \( X = \{ x \in \{0,1\}^6 : 45x_1 + 46x_2 + 79x_3 + 54x_4 + 53x_5 + 125x_6 \leq 178 \} \), fractional \( x^* = (0,0,3/4,1/2,1,0) \)
- Solve:
  \[
  \beta = \min z_1 + z_2 + \frac{1}{4} z_3 + \frac{1}{2} z_4 + 0z_5 + z_6 \\
  \text{s.t.} \quad 45z_1 + 46z_2 + 79z_3 + 54z_4 + 53z_5 + 125z_6 > 178 \\
  \quad z \in \{0,1\}^6
  \]
  \( z^* = (0,0,1,1,1,0) \) with \( \beta = 3/4 \)
- Conclude that \( x_3 + x_4 + x_5 \leq 2 \) is a cut for \( x^* \).

  Why is this useful? Because we can try to solve this “auxiliary problem” heuristically and give up if it is too hard.
  Can also combine with lifting (complete for 0/1 knapsack sets)
• We’ve seen two families of cuts:
  – Chvatal-Gomory cuts
  – Cover cuts

• There are many others:
  – Flow cover
  – Mixed integer rounding
  – Odd hole
  – ...

• How can we use these?

The Branch-and-Cut Method
Reminder

• **Valid inequality**: an inequality satisfied by all feasible solutions
• **Cut**: a valid inequality that is not part of the current formulation

The Branch-and-Cut method

• Generalizes Branch-and-Bound.
• If cannot prune a node, can try to find a cut.
• Strengthen the formulation; re-solve the LP, may now be able to prune by bound.
• If no cuts are found, then branch.
The Branch-and-Cut method

• Branch-and-Bound + **cuts generated throughout the tree.**
• Cuts generated at root are “global,” cuts at subproblems “local.” Place into a “cut pool.”
• **Goal:** reduce the number of search nodes by improving **bounds**

Example: Branch and Cut

• Generalized assignment problem
• 10 firms, 5 distinct goods (≈100 units each)
• Each firm demands a particular quantity of each good
• Goal: maximize total value
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<tr>
<th>Firms</th>
<th>Goods</th>
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</tr>
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<td>10</td>
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</table>

**Supply**

|       | 91   | 87   | 109  | 88  | 64  |

### Assignment problem

- Firms $i$, goods $j$ with supply $b_j$
- $c_{ij}$ value to $i$ for allocation of $a_{ij}$ units of good $j$
- $x_{ij}=1$ if $a_{ij}$ units of good $j$ assigned to agent $i$

\[
\begin{align*}
\text{max} & \quad \sum_{i} \sum_{j} c_{ij} x_{ij} \\
\text{s.t.} & \quad \sum_{j} x_{ij} \leq 1 \quad \text{for all } i = 1, \ldots, m \\
& \quad \sum_{i} a_{ij} x_{ij} \leq b_{j} \quad \text{for all } j = 1, \ldots, n \\
& \quad x_{ij} \in \{0, 1\}
\end{align*}
\]

Has “0/1 knapsack set” structure. Can use Cover cuts.
- Generate lifted cover cuts (global and local)
Historical Context

• Use of cutting planes as a practical tool for solving IPs had been abandoned in ‘60s and ‘70s
• Revisited in early ‘70s following NP-completeness theory (interest in problem structure and facets.)
• Balas, Wolsey, Johnson, Padberg in mid ‘70s developed cover inequalities and “lifting”
• Branch-and-cut developed in early ‘80s, and introduced into commercial solvers in 2000s
• Has provided an order of magnitude improvement in solve times!

Other important components

• Pre-processing
  • Primal heuristics
  • SOS branching
(1 of 3) Preprocessing

• Detect and eliminate redundant constraints and variables, tighten bounds where possible

Consider LP: (Wolsey pp.104)

max \(2x_1 + x_2 - x_3\)

\[
\begin{align*}
5x_1 - 2x_2 + 8x_3 &\leq 15 \quad \text{(a)} \\
8x_1 + 3x_2 - x_3 &\geq 9 \quad \text{(b)} \\
x_1 + x_2 + x_3 &\leq 6 \quad \text{(c)} \\
0 &\leq x_1 \leq 3; \quad 0 \leq x_2 \leq 1; \quad 1 \leq x_3
\end{align*}
\]

• Isolate \(x_1\) in (a) to obtain \(5x_1 \leq 15 + 2 - 8 = 9\), and so \(x_1 \leq 9/5\).
• Isolate \(x_3\) in (a) to obtain \(x_3 \leq 17/8\). Not useful to isolate \(x_2\).
• Can also isolate \(x_1\) in (b); obtain \(x_1 \geq 7/8\).

Repeat: improve bound on \(x_3\) from (a) by using \(x_1 \geq 7/8\); obtain \(x_3 \leq 101/64\)

• Find constraint (c) is redundant, using \(x_1 \leq 9/5\) and \(x_3 \leq 101/64\), together with \(x_2 \leq 1\).
“Fixing” values of variables

The modified LP is:

• \[ \text{max } 2x_1 + x_2 - x_3 \]
  \[\text{s.t. } 5x_1 - 2x_2 + 8x_3 \leq 15 \]
  \[8x_1 + 3x_2 - x_3 \geq 9 \]
  \[7/8 \leq x_1 \leq 9/5; \ 0 \leq x_2 \leq 1; \ 1 \leq x_3 \leq 101/64 \]

• **Fixing:** In optimal solution, have \( x_2 = 1; \) and \( x_3 = 1. \)
• Why? \( x_2 \) has positive coeff in obj, and increasing makes constraints less tight. Similarly, best to decrease \( x_3. \)

• Obtain problem: \( \text{max}\{2x_1 : 7/8 \leq x_1 \leq 9/5}\}. \) Easy to solve!

Other important components

• Pre-processing
• Primal heuristics
• SOS branching
(2 of 3) Primal heuristics

• At a fractional search node, use a heuristic approach to try to find an integral solution

• For example, round the solution or do a directed “neighborhood search.”

• May be able to improve incumbent.

Other important components

• Pre-processing
• Primal heuristics
• SOS branching
(3 of 3) SOS branching

- Consider constraints $\sum_{j=1}^{n} x_j = 1$, $x_j \in \{0,1\}$
- Typical to branch on fractional $x_k^*$:

```
unbalanced
\[ x_k \leq 0 \quad \text{n-1 solutions} \quad x_k \geq 1 \quad \text{1 solution} \]
```

- “Special-ordered sets” branching. Pick the first $k$ s.t. $\sum_{j=1}^{k} x_j^* \geq \frac{1}{2}$

```
balanced
\[ \sum_{j=1}^{k} x_j \leq 0 \quad \sum_{j=k+1}^{n} x_j \leq 0 \quad (\text{or } \sum_{j=1}^{k} x_j \geq 1) \]
```

Example: $x^*=(0,0.2,0.4,0,0.4)$. Branch $x_1 + x_2 + x_3 \leq 0$, $x_4 + x_5 \leq 0$

Summary: Advanced IP solving

- MIPs are a lot harder to solve than LPs
- But IP technology is very sophisticated:
  - Dual simplex pivots
  - Tight formulations
  - Automated cut generation
  - Primal heuristics
  - Node and variable selection heuristics
- Problems with 100,000s variables and 10,000s constraints can be solved in commercially viable times