Lecture 15: Cutting plane methods

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Lesson Plan

• Cut generation and the separation problem
• Cutting plane methods
• Chvátal-Gomory cuts
• Gomory’s cutting plane algorithm

Textbook Reading: 8.4
Convex Hull

• **Definition.** Given set $X \subseteq \mathbb{Z}^n$, the convex hull of $X = \{x^1, \ldots, x^t\}$ is $\text{conv}(X) = \{x : x = \sum_{k=1}^{t} \lambda_k x^k, \sum_{k=1}^{t} \lambda_k = 1, \lambda_k \geq 0 \text{ for all } k\}$

• **Prop.** $\text{conv}(X)$ is a polyhedron.

• **Prop.** Extreme points of $\text{conv}(X)$ all lie in $X$.

• **Prop.** Can solve IP via solving LP on $\text{conv}(X)$.

**Challenge:** May need exponential number of inequalities to describe $\text{conv}(X)$.

Cut Generation

• Q: What else can we do to improve the strength of formulations?

• A: *Automatically generate new inequalities (“cuts”) that approximate the convex hull.*

• Why this might be useful:
  – improve branch-and-bound (stronger formulation, and thus improved bounds)
  – provides a completely new way to solve IPs
Definition. An inequality $a^T x \leq b$ is a valid inequality for set $X \subseteq \mathbb{Z}^n$ if $a^T x \leq b$ for all $x \in X$.

“keeps all integer solutions”

Defn. Given fractional solution $x^*$, the separation problem is “is $x^* \in \text{conv}(X)$?” If NO, find a valid inequality that is violated by $x^*$.

Defn. A cut is a valid inequality that separates the current fractional solution $x^*$.
• **Definition.** An inequality $a^T x \leq b$ is a **valid inequality** for set $X \subseteq \mathbb{R}^n$ if $a^T x \leq b$ for all $x \in X$.

• **Defn.** A **cut** is a valid inequality that **separates** the current fractional solution $x^*$.

![Diagram of valid and invalid cuts](image)

**Cut strength**

• **Definition.** Cut $c$ is stronger than cut $c'$ if $z^{\text{LP}} < z^{\text{LP}'}$, where LP includes $c$ and LP' includes $c'$.

![Diagram of stronger cuts](image)
The Cutting-Plane Method

• Step 1: Solve LPR. Get $x^*$.  
• Step 2: If $x^*$ integral, **stop**. Else, find a valid inequality that excludes $x^*$ (a “cut”) 
• Step 3: Go to Step 1.

➤ keep strengthening the formulation until the IP is solved

Questions

• How to generate **strong cuts**, and quickly?  
• Will a cutting-plane algorithm always **terminate** with the **optimal IP solution**?

• For warm-up, let’s look at examples of valid inequalities.
Example 1

• X = \{x \in \{0,1\}^5 : 3x_1−4x_2+2x_3−3x_4+x_5 \leq -2\}

• First: If \(x_2=x_4=0\), the LHS \geq 0, and the solution is infeasible.
  – By integrality, a valid inequality is \(x_2+x_4 \geq 1\).
  – Does not remove any \(x \in X\), so valid. Also: removes fractional solns eg., \(x=(0,1/3, 0, 1/3, 0)\).

• Second: If \(x_2=0\) but \(x_1=1\), the LHS \geq 0, and the solution is infeasible.
  – By integrality, a valid inequality is \(x_1 \leq x_2\).
  – Does not remove any \(x \in X\), so valid. Also: removes fractional solns e.g., \(x=(2/6,1/6, 0, 1, 0)\).

Example 2

• X = \{(x,y) : x \leq 9999y, 0 \leq x \leq 5, x \in Z, y \in \{0,1\} \}\}

• X = \{(0,0),(0,1),(1,1),(2,1),(3,1),(4,1),(5,1)\}

• A valid inequality is \(x \leq 5y\).
  – Does not remove any solutions in X, so valid.
  – Also: removes fractional solutions such as \((1,0.1)\).
Example 3
• $X = \{(x,y) : x \leq 10y, 0 \leq x \leq 14, y \in \mathbb{Z}_{\geq 0}\}$
• $x \leq 6 + 4y$ is valid. Doesn’t remove any solns in $X$

Example 4 (Chvátal-Gomory inequality)
• Consider $X = P \cap \mathbb{Z}^2$, where $P$ is given by
  \begin{align*}
  7x_1 - 2x_2 &\leq 14 \\
  x_2 &\leq 3 \\
  2x_1 - 2x_2 &\leq 3 \\
  x &\geq 0
  \end{align*}

  Notes:
  (a) Important that $X$ is non-negative integers
  (b) Applies to "≤" inequalities
  (c) The result of steps (1) and (2) is implied by $P$ and is weaker. But the integrality on the LHS allows step 3 to tighten!

• Valid to form a non-neg, linear combination of inequalities. Eg., multipliers $u=(2/7, 37/63, 0)$. Obtain:
  $$2x_1 + 1/63x_2 \leq 121/21$$

• Since $x \geq 0$, valid to round coeffs on LHS down:
  $$2x_1 + 0x_2 \leq 121/21$$

• Because LHS is integral for all $x \in X$, valid to round RHS down to nearest integer. Obtain:
  $$2x_1 + 0x_2 \leq 5$$
General Approach to CG Inequality

\[ X = P \cap Z^n, \quad P = \{ x \in R^n_{\geq 0} : Ax \leq b \}, \text{ some } u \in R^m_{\geq 0} \]

Three steps:

(i) Combine rows of A with nonnegative weights \( u \). This is valid for \( X \). Obtain an \( \leq \) inequality.

(ii) Take the floor of coefficients on LHS. This is a valid inequality for \( X \) because \( x \geq 0 \).

(iii) LHS has integer value \( \Rightarrow \) valid to take the floor of the RHS. [The LHS has integer values because coeffs and \( x \) are integer.]
• **Theorem.** Given any fractional, extreme point $x^*$ of $P$, there exists multipliers $u \geq 0$ s.t. the CG inequality is a cut (for $x^*$).

$\Rightarrow$ **CG cuts are complete for IP!!** In principle we can solve IPs by repeated CG cuts.

But:

• How many CG cuts do we need to generate?
• How do we generate the cuts?

**Answer: Use Gomory’s algorithm**

• CG inequalities are valid for any integer program with “≤” and non-neg, integer decision variables.
• Gomory’s algorithm uses CG inequalities in a particular way. It can only be applied to an integer program with **integer coeffs** and **integer RHS’s.**

(WLOG for rational problem: achieve by rescaling)

• It generates CG cuts, and provably converges to the optimal IP solution.
• Uses the simplex tableau, generates a cut from any row with a fractional RHS value.
• No fractional rows left $\Rightarrow$ an integer solution!
Gomory's cutting plane algorithm

- “Outline of an algorithm for Integer solutions to linear programs,”
- “An algorithm for Integer solutions to Linear programs,”
- “Edmonds polytopes and a hierarchy of combinatorial problems,”

Example (Gomory’s algorithm)

- For an IP with $A \in \mathbb{Z}^{mn}$ and $b \in \mathbb{Z}^m$.
- For example:
  \[
  z = \max 4x_1 - x_2 \\
  \text{s.t.} \quad 7x_1 - 2x_2 \leq 14 \\
  \quad x_2 \leq 3 \\
  \quad 2x_1 - 2x_2 \leq 3 \\
  \quad x_1, x_2 \geq 0, \text{ integer}
  \]
- Introduce slack variables $x_3, x_4$ and $x_5$.
  - $A$ and $b$ integer: therefore, we can insist that slack variables are non-negative, integer.

(Need integer coefficients, RHS. WLOG for rational problem: achieve by rescaling.)
Generating a CG cut

Optimal tableau:

- \( z = \frac{4}{7} x_3 + \frac{1}{7} x_4 = \frac{59}{7} \)
- \( x_1 = \frac{1}{7} x_3 + \frac{2}{7} x_4 = \frac{20}{7} \)
- \( x_2 = \frac{1}{7} x_3 + x_4 = 3 \)
- \( -\frac{2}{7} x_3 + \frac{10}{7} x_4 + x_5 = \frac{23}{7} \)

- \( B = \{1, 2, 5\} \) and \( x^* = (\frac{20}{7}, 3, 0, 0, \frac{23}{7}) \)

- Form a CG-cut from any row with a fractional RHS.
  - “u” vector puts 1 on this row, 0 on other rows.
  - apply CG procedure to the “\( \leq \)” inequality implied by the equality (CG is defined for “\( \leq \)” inequalities).

- Suppose we choose \( x_1 \) row. The CG cut is:
  \( x_1 + 0x_3 + 0x_4 \leq 2 \)
• \( z + \frac{4}{7} x_3 + \frac{1}{7} x_4 = \frac{59}{7} \)  
  \( x_1 + \frac{1}{7} x_3 + \frac{2}{7} x_4 = \frac{20}{7} \)  
  \( x_2 + x_4 = 3 \)  
  \(-\frac{2}{7} x_3 + \frac{10}{7} x_4 + x_5 = \frac{23}{7} \)  

• Add CG cut from \( x_1 \) row (convenient to negate):
  \[-x_1 \geq -2 \]  
  
• Will want to isolate current basis. (1) + (a):
  \[ \frac{1}{7} x_3 + \frac{2}{7} x_4 \geq \frac{6}{7} \]  

• Add non-neg, integer excess var \( x_6 \geq 0 \):
  \[ \frac{1}{7} x_3 + \frac{2}{7} x_4 - x_6 = \frac{6}{7} \]  

• Note: \( x_6 \) integer because LHS-RHS in (a) is integer, also true for (b) since (1) is an equality.

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Optimal tableau:

<table>
<thead>
<tr>
<th></th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( 2x_1 - 2x_2 \leq 3 )</td>
<td>( 7x_1 - 2x_2 \leq 14 )</td>
<td>( x_1 \leq 2 )</td>
<td>(2,1/2)</td>
<td>(20/7,3)</td>
</tr>
<tr>
<td>2</td>
<td>( x_2 \leq 3 )</td>
<td>( x_1 \leq 2 )</td>
<td>( 2x_1 - 2x_2 \leq 3 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
• Add (c) and re-optimize.
• New optimal tableau:

\[
\begin{align*}
 z &+ \frac{1}{2}x_5 + 3x_6 = 15/2 \\
 x_1 &+ x_6 = 2 \\
x_2 &- \frac{1}{2}x_5 + x_6 = \frac{1}{2} \\
x_3 &- x_5 - 5x_6 = 1 \\
x_4 &+ \frac{1}{2}x_5 + 6x_6 = 5/2
\end{align*}
\]

• \( B=\{1,2,3,4\} \) and \( x^*=(2, 1/2, 1, 5/2, 0, 0) \)

• Add another CG cut. Use a general rule to go from fractional tableau to a new cut.

**General Rule for Deriving a Cut**

For row \( i \) with fractional RHS, the CG cut is

\[
\sum_{j \in B^*} (\bar{a}_{ij} - a_{ij}) x_j \geq \bar{b}_i - b_i
\]

**Example 1:**

- \( z + 4/7 x_3 + 1/7 x_4 = 59/7 \)
- \( x_1 + 1/7x_3 + 2/7 x_4 = 20/7 \)
- \( x_2 + x_4 = 3 \)
- \( -2/7 x_3 + 10/7x_4 + x_5 = 23/7 \)

• \( B=\{1, 2, 5\} \)
• Cut \( 1/7x_3 + 2/7x_4 \geq 6/7 \)
General Rule for Deriving a Cut

- For row $i$ with fractional RHS, the CG cut is
  \[
  \sum_{j \in B'} (\bar{a}_{ij} - \bar{a}_{ij}) x_j \geq \bar{b}_i - \bar{b}_j
  \]

Example 2:
- $B=\{1, 2, 3, 4\}$
- $\text{Cut } \frac{1}{2}x_5 \geq \frac{1}{2}$ ($\frac{1}{2} = -\frac{1}{2} - (-\frac{1}{2}) = -\frac{1}{2} - (-1) = \frac{1}{2}$)

- Add (c) and re-optimize.
- New optimal tableau:
  \[
  \begin{align*}
  z & \quad +\frac{1}{2}x_5 + 3x_6 = 15/2 \\
  x_1 & \quad +x_6 = 2 \\
  x_2 & \quad -\frac{1}{2}x_5 + x_6 = \frac{1}{2} \\
  x_3 & \quad -x_5 - 5x_6 = 1 \\
  x_4 & \quad +\frac{1}{2}x_5 + 6x_6 = 5/2
  \end{align*}
  \]
- $B=\{1, 2, 3, 4\}$ and $x^*=(2, 1/2, 1, 5/2, 0, 0)$
- Add CG cut $\frac{1}{2}x_5 \geq \frac{1}{2}$
- Bring in integer, non-neg excess var $x_7$.
  \[
  \frac{1}{2}x_5 - x_7 = \frac{1}{2}
  \]
• Add (d) and re-optimize. New tableau:

\[
\begin{align*}
    & z + 3x_6 + x_7 = 7 \\
    & x_1 + x_6 = 2 \\
    & x_2 + x_6 - x_7 = 1 \\
    & x_3 - 5x_6 - 2x_7 = 2 \\
    & x_4 + 6x_6 + x_7 = 2 \\
    & x_5 - 2x_7 = 1
\end{align*}
\]

• \(x^*=(2, 1, 2, 2, 1, 0, 0)\)

• Integral! \((x_1, x_2)=(2,1)\) solves the original IP.


• Map the cuts back to the \((x_1, x_2)\) space
• First cut: \(x_1 \leq 2\)
• Second cut: \(\frac{1}{2}x_5 \geq \frac{1}{2}\). Substituting for \(x_5=3-2x_1+2x_2\), rearrange and obtain \(x_1-x_2 \leq 1\).
Summary: Gomory’s algorithm

- **Repeat:**
  - Solve LP.
  - If integral, STOP. Else, generate a CG-cut from any row with fractional RHS.

- **Thm.** Gomory’s algorithm will solve an IP with *integer coefficients and integer RHS values.*
Example (Gomory’s algorithm)

• Consider the IP
  
  \[
  \begin{align*}
  z &= \text{max } 4x_1 - x_2 \\
  7x_1 - 2x_2 &\leq 14 \\
  x_2 &\leq 3 \\
  2x_1 - 2x_2 &\leq 3 \\
  x_1, x_2 &\geq 0, \text{ integer}
  \end{align*}
  \]

• Initial optimal tableau:

  \[
  \begin{align*}
  z + \frac{4}{7}x_3 + \frac{1}{7}x_4 &= \frac{59}{7} \\
  x_1 + \frac{1}{7}x_3 + \frac{2}{7}x_4 &= \frac{20}{7} \\
  x_2 + x_4 &= 3 \\
  -\frac{2}{7}x_3 + \frac{10}{7}x_4 + x_5 &= \frac{23}{7}
  \end{align*}
  \]

  B={1, 2, 5}

  • Add cut \( \frac{1}{7}x_3 + 2/7x_4 \geq 6/7 \)
  
  • (Bring in integer excess variable \( x_6 \))
• New optimal tableau:
  \[ z + \frac{1}{2}x_5 + 3x_6 = 15/2 \]
  \[ x_1 + x_6 = 2 \]
  \[ x_2 - \frac{1}{2}x_5 + x_6 = \frac{1}{2} \]
  \[ x_3 - x_5 - 5x_6 = 1 \]
  \[ x_4 + \frac{1}{2}x_5 + 6x_6 = 5/2 \]
  \[ B = \{1, 2, 3, 4\} \text{ and } x^* = (2, 1/2, 1, 5/2, 0, 0) \]
  \[ \text{Add cut } \frac{1}{2}x_5 \geq \frac{1}{2} \]
  \[ \text{(Bring in integer excess variable } x_7) \]

• Final optimal tableau:
  \[ z + 3x_6 + x_7 = 7 \]
  \[ x_1 + x_6 = 2 \]
  \[ x_2 + x_6 - x_7 = 1 \]
  \[ x_3 - 5x_6 - 2x_7 = 2 \]
  \[ x_4 + 6x_6 + x_7 = 2 \]
  \[ x_5 - x_7 = 1 \]
  \[ x^* = (2, 1, 2, 2, 1, 0, 0) \]
Gomory’s algorithm

• Not of practical interest, but transformative method!
• Can solve IPs without branch and bound, and inspired effort into identifying families of strong cuts.
• With some work, can extend to solve mixed IPs

Summary: Cutting plane method

• Cuts are valid inequalities that separate the current optimal, fractional solution
• The cutting-plane method
• Chvatal-Gomory cuts for IPs (they are complete!)
• Gomory’s cutting plane method
  – Solves IPs without branch and bound
  – Works on IPs with integer coefficients and integer RHS.