AM 121: Intro to Optimization Models and Methods

Lecture 15: Cutting plane methods

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Lesson Plan

• Cuts and the separation problem
• Cutting plane methods
• Chvatal-Gomory cuts
• Gomory’s cutting plane algorithm

Textbook Reading: 8.4
Recall: Convex Hull

- **Definition.** Given set $X \subseteq \mathbb{Z}^n$, the convex hull of $X=\{x^1, \ldots, x^t\}$ is $\text{conv}(X)=\{x: x=\sum_{k=1}^{t} \lambda_k x^k, \sum_{k=1}^{t} \lambda_k=1, \lambda_k \geq 0 \text{ for all } k\}$

(Defn also generalizes to allow $X \subseteq \mathbb{R}^n$)

- **Prop.** Extreme points of $\text{conv}(X)$ all lie in $X$.

- **Prop.** Can solve IP via solving LP on $\text{conv}(X)$.

**Challenge:** May need exponential number of inequalities to describe $\text{conv}(X)$.

Cut Generation

- **Q:** What else can we do to improve the strength of formulations?

- **A:** Automatically generate new inequalities ("cuts") that try approximate the convex hull.

- **Why this might be useful:**
  - improve speed of branch-and-bound (stronger formulation, improved bounds)
  - allow a completely new way to solve IPs
• **Definition.** An inequality $a^T x \leq b$ is a **valid inequality** for set $X \subseteq \mathbb{R}^n$ if $a^T x \leq b$ for all $x \in X$.

- **Defn.** Given set $X \subseteq \mathbb{R}^n$ and $x^* \in \mathbb{R}^n$, the **separation problem** is “is $x^* \in \text{conv}(X)$?” If NO, find a valid inequality that is violated by $x^*$.
- **Defn.** A **cut** is a valid inequality that separates the current fractional solution $x^*$.
Cut strength

- **Definition.** Cut $c$ is stronger than cut $c'$ if $z^{LP} < z^{LP'}$, where these are the LP upper-bounds of the LPRs.

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The Cutting Plane Method

- Step 1: Solve LPR. Get $x^*$.  
- Step 2: If $x^*$ integral, **stop**. Else, find a valid inequality that excludes $x^*$ (a “cut”)  
- Step 3: Go to Step 1.

⇒ work to strengthen the formulation until the IP is solved
Questions

• How to generate strong cuts, and quickly?
• Will a cutting plane algorithm always terminate with the optimal IP solution?

• For warm-up, let’s look at examples of valid inequalities.

Example 1

• $X = \{ x \in \{0, 1\}^5 : 3x_1 – 4x_2 + 2x_3 – 3x_4 + x_5 \leq -2 \}$
• If $x_2 = x_4 = 0$, the LHS must be $\geq 0$, and the solution is infeasible.
  – By integrality, a valid inequality is $x_2 + x_4 \geq 1$.
  – Does not remove any $x \in X$. Removes fractional $x = (0, 1/3, 0, 1/3, 0)$.
• If $x_1 = 1$ and $x_2 = 0$, the LHS must be $\geq 0$, and the solution is infeasible.
  – By integrality, a valid inequality is $x_1 \leq x_2$.
  – Does not remove any $x \in X$. Removes fractional $x = (2/6, 1/6, 0, 1, 0)$. 
Example 2

• $X = \{(x,y) : x \leq 9999y, \ 0 \leq x \leq 5, \ x \in \mathbb{Z}, \ y \in \{0,1\}\}$

• Feasible set is $X = \{(0,0), (0,1), (1,1), (2,1), \ldots, (5,1)\}$

• A valid inequality is $x \leq 5y$.
  – Does not remove any solutions in $X$.
  – Removes fractional solutions such as $(1,0.1)$.

Example 3

• $X = \{(x,y) : x \leq 10y, \ 0 \leq x \leq 14, \ y \in \mathbb{Z}_{\geq 0}\}$

• $x \leq 6 + 4y$ is valid.
Example 4 (Chvátal-Gomory inequality)

• Consider \( X = P \cap \mathbb{Z}^2 \), where \( P \) is given by:
  \[
  7x_1 - 2x_2 \leq 14 \\
  x_2 \leq 3 \\
  2x_1 - 2x_2 \leq 3 \\
  x \geq 0
  \]

• Form a linear combination of inequalities. Suppose we adopt multipliers \( u = (2/7, 37/63, 0) \geq 0 \). Obtain:
  \[
  2x_1 + 1/63x_2 \leq 121/21
  \]

• Valid to round coeff’s on LHS down to nearest integer:
  \[
  2x_1 + 0x_2 \leq 121/21
  \]

• Because LHS is \textbf{integral} for all \( x \in X \), valid to round RHS down to nearest integer. Obtain:
  \[
  2x_1 + 0x_2 \leq 5
  \]

Cuts off \((20/7, 3)\), which is optimal fractional solution.

General Approach to CG Inequality

\( X = P \cap \mathbb{Z}^n \), \( P = \{ x \in \mathbb{R}^n_{\geq 0} : Ax \leq b \} \), some \( u \in \mathbb{R}^m_{\geq 0} \)

Three steps:
(i) the inequality formed by combining rows of \( A \) with nonnegative weights \( u \) is valid for \( X \)
(ii) the inequality formed from floor of coefficients on the LHS is valid for \( X \) because \( x \geq 0 \)
(iii) the inequality formed from floor of RHS value is valid for \( X \) because \( x \in X \) is integer, and LHS is integer
• **Theorem.** Given any fractional, extreme point $x^*$ of $P$, there exists multipliers $u \geq 0$ s.t. the CG inequality is valid and cuts off $x^*$.

$=>$ CG cuts are complete for IP; in principle we can solve IPs by repeated CG cuts.

But:
• How many CG cuts do we need to generate?
• **How do we generate the right cuts?**

**Answer: Use Gomory’s algorithm**

• For an IP with integer coefficients and integer right-hand side values.
• Generates CG cuts, and sure to converge to the optimal IP feasible solution.
• Algorithm uses the simplex tableau, generates a cut from **one** inequality; in particular, an inequality corresponding to any row in the tableau with a fractional RHS value.
• When there are no such fractional row left, we have a $\{0,1\}$ solution.
Gomory's cutting plane algorithm

• “Outline of an algorithm for Integer solutions to linear programs,”
  R.E.Gomory, Bulletin of the American Mathematical Society 64, 275-278 (1958)
• “An algorithm for Integer solutions to Linear programs,”
• “Edmonds polytopes and a hierarchy of combinatorial problems,”
  V. Chvatal, Discr. Math. 4, 305–337 (1973)

Example (Gomory’s algorithm)

• For an IP with $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$.
• For example:

$$z = \max 4x_1 - x_2$$

s.t. 

$$7x_1 - 2x_2 \leq 14$$

$$x_2 \leq 3$$

$$2x_1 - 2x_2 \leq 3$$

$$x_1, \quad x_2 \geq 0, \text{ integer}$$

• Introduce slack variables $x_3, x_4$ and $x_5$.
  – $A$ and $b$ integer: therefore, we can insist that slack variables are non-negative, integer.
Generating a CG cut

• $z + \frac{4}{7} x_3 + \frac{1}{7} x_4 = \frac{59}{7}$
  
  $x_1 + \frac{1}{7} x_3 + \frac{2}{7} x_4 = \frac{20}{7}$
  
  $x_2 + x_4 = 3$
  
  $-\frac{2}{7} x_3 + \frac{10}{7} x_4 + x_5 = \frac{23}{7}$

• $B=\{1, 2, 5\}$ and $x^*(20/7, 3, 0, 0, 23/7)$

• Form a CG-cut from a single row. Choose any row with a fractional RHS.
  – the “$u$” vector is implicit in the simplex algorithm.
  – it is allowed to apply the CG procedure to an equality, since if $a^T x = b$, then $a^T x \leq b$.

• Suppose we choose row corresponding to basic variable $x_1$.

• The CG cut is as follows (and it cuts off $x^*$):
  
  $x_1 + 0x_3 + 0x_4 \leq 2$

• $z + \frac{4}{7} x_3 + \frac{1}{7} x_4 = \frac{59}{7}$
  
  $x_1 + \frac{1}{7} x_3 + \frac{2}{7} x_4 = \frac{20}{7}$ (1)
  
  $x_2 + x_4 = 3$
  
  $-\frac{2}{7} x_3 + \frac{10}{7} x_4 + x_5 = \frac{23}{7}$

• Want to re-solve the LP with new cut:
  
  $-x_1 - 0x_3 - 0x_4 \geq -2$ (*)

• Useful to reformulate in terms of non-basic vars only

• Add (1) and rearrange:
  
  $\frac{1}{7} x_3 + \frac{2}{7} x_4 \geq \frac{6}{7}$ (**) 

• Note the difference between the LHS and RHS in (*) is integral; therefore, also true for (**)

• Can introduce excess variable $x_6 \geq 0$, and insist that it is integral.
The general rule for the CG cut on row i with fractional RHS, is \( \sum_{j \in B} \pi_j x_j \geq \pi_0 \); with
\[
\pi_i = \bar{a}_{ij} - \left[ \bar{a}_{ij} \right] \\
\pi_o = \bar{b}_i - \left| \bar{b}_i \right|
\]

- \( z + \frac{4}{7} x_3 + \frac{1}{7} x_4 = \frac{59}{7} \)
- \( x_1 + \frac{1}{7}x_3 + \frac{2}{7} x_4 = \frac{20}{7} \)
- \( x_2 + x_4 = 3 \)
- \( -\frac{2}{7} x_3 + \frac{10}{7} x_4 + x_5 = \frac{23}{7} \)

Let's check on row 1:
- \( \pi_3 = \frac{1}{7} - 0 = \frac{1}{7} \)
- \( \pi_4 = \frac{2}{7} - 0 = \frac{2}{7} \)
- \( \pi_0 = \frac{20}{7} - 2 = \frac{6}{7} \)
\[
1/7x_3 + 2/7x_4 \geq 6/7 \text{ (**)}
\]

Note: all coefficients will be \( \geq 0 \)

Add (**) and re-optimize.

New optimal tableau:
\[
z + \frac{1}{2} x_5 + 3 x_6 = \frac{15}{2} \\
x_1 + x_6 = 2 \\
x_2 - \frac{1}{2} x_5 + x_6 = \frac{1}{2} \\
x_3 - x_5 - 5 x_6 = 1 \\
x_4 + \frac{1}{2} x_5 + 6 x_6 = \frac{5}{2}
\]

B={1,2,3,4} and \( x^* = (2, \frac{1}{2}, 1, \frac{5}{2}, 0, 0) \)

Now arbitrarily choose to add the CG cut for the row corresponding to basic var \( x_2 \)

The CG cut is \( \frac{1}{2} x_5 \geq \frac{1}{2} \). Add integer, non-neg excess var \( x_7 = \frac{1}{2} x_5 - \frac{1}{2} \).

Re-solve
• Adding constraint, re-optimize. New tableau:

\[
\begin{align*}
    z & + 3x_6 + x_7 = 7 \\
    x_1 & + x_6 = 2 \\
    x_2 & + x_6 - x_7 = 1 \\
    x_3 & - 5x_6 - 2x_7 = 2 \\
    x_4 & + 6x_6 + x_7 = 2 \\
    x_5 & - x_7 = 1
\end{align*}
\]

• \( x^* = (2, 1, 2, 2, 1, 0, 0) \)

• Integral! \((x_1, x_2) = (2,1)\) solves the original IP.


• Let’s map the cuts back to the \((x_1, x_2)\) space
• First cut: \( x_1 \leq 2 \)
• Second cut: \( \frac{1}{2}x_5 \geq \frac{1}{2} \). Substituting for \( x_5 = 3 - 2x_1 + 2x_2 \), rearrange and obtain \( x_1 - x_2 \leq 1 \).
Summary: Gomory’s algorithm

- **Repeat:**
  - Solve LP.
  - If integral, STOP. Else, generate a CG-cut from **any** row with a fractional RHS.

- **Thm.** Gomory’s algorithm will solve an IP with integer coefficients and integer RHS values, converging after a finite number of iterations.

- Not of practical interest, but transformative method.
- Can solve IPs without branch and bound! Inspired effort into identifying families of cuts.
Summary

- Cuts the separation problem
- Cutting plane method
- Chvatal-Gomory cuts
- Gomory’s cutting plane method
  - Solves IPs without branch and bound
  - Works on IPs with integer coefficients and integer RHS (also extends to IPs with rational coefficients, rational RHS values.)