Lecture 14: Branch and Bound (II)

Yiling Chen
SEAS

Lesson Plan

- Branch and Bound review
- Node selection
- Branching decision
- Formulation strength
- CPLEX and “Cut generation”
Review

\[
\begin{align*}
\text{max } & 5x_1 + 8x_2 \\
\text{s.t. } & x_1 + x_2 \leq 6 \\
& 5x_1 + 9x_2 \leq 45 \\
& x_1, x_2 \geq 0, \text{ integer}
\end{align*}
\]
Branch and Bound Method

- Maintain a list of open subproblems, an incumbent $\mathbf{x}$ with value $z$, and an upper bound $\bar{z}$

- **Node selection decision**: pick open subproblem and solve LP relaxation

- **Branching decision**: if can’t fathom the node, then pick a fractional variable and branch

- Continue until all open subproblems are fathomed, or “optimality gap” is acceptable.
**Node selection** Decision

- **Depth-first search** (solve a node just generated)
  - Find an integer solution quickly. This way, open subproblems can be “fathomed by bound.”
  - Can obtain next LP solution via **dual simplex pivots**, since branching adds or modifies a constraint.

- **Best-bound first** (solve node $k$ with highest LPR $Z_k$.)
  (Upper bound on subproblem is inherited from parent)
  - Never solve a subproblem with an upper-bound less than the value of **optimal** integer solution.
  - Improve upper bounds quickly, try to prove optimality of current incumbent.

- In practice: initial DFS (“diving”) followed by a mix of best-bound and DFS is effective.

DFS will solve one of $S^5$ or $S^6$
“Best-bound first” will solve $S^4$
(since $41 > 40 \frac{5}{9}$)
Example: Dual Pivots

• Optimal tableau for LPR of IP is
  \[
  \begin{align*}
  z + 1.25x_3 + 0.75x_4 &= 41.25 \\
  x_1 + 2.25x_3 - 0.25x_4 &= 2.25 \\
  x_2 - 1.25x_3 + 0.25x_4 &= 3.75
  \end{align*}
  \]
  \(1\)

• Consider LPR of \(S^2 = S^0 \cap \{x_2 \geq 4\}\)

• Introduce excess variable \(x_5 \geq 0\), and write \(x_2 \geq 4\) as
  \[
  x_2 - x_5 = 4 \quad (2)
  \]

• Establish basis \(B = \{x_1, x_2, x_5\}\) by \((2)^\prime = (1) - (2)\)
  \[
  -1.25x_3 + 0.25x_4 + x_5 = -0.25 \quad (2^\prime)
  \]

Using Dual Pivots in BnB

• Initial tableau for LPR of \(S^2\) is dual feasible:
  \[
  \begin{align*}
  z + 1.25x_3 + 0.75x_4 &= 41.25 \\
  x_1 + 2.25x_3 - 0.25x_4 &= 2.25 \\
  x_2 - 1.25x_3 + 0.25x_4 &= 3.75
  \end{align*}
  \]
  \[
  -1.25x_3 + 0.25x_4 + x_5 = -0.25
  \]

• Dual Pivot (\(x_5\) out, \(x_3\) in). Get:
  \[
  \begin{align*}
  z + x_4 + x_5 &= 41 \\
  x_1 + 0.2x_4 + 1.8x_5 &= 1.8 \\
  x_2 - x_5 &= 4 \\
  x_3 - 0.2x_4 - 0.8x_5 &= 0.2
  \end{align*}
  \]

• \(B = \{x_1, x_2, x_3\}\), \(x_1^* = 1.8, x_2^* = 4, z^* = 41\)

• **One pivot!** In general will take more than one pivot, but can often find new LP solution quickly.
Aside: Weak Dual Pairs

• Consider
  Primal: \( \max \{ c(x) : x \in X \} \)
  Dual: \( \min \{ w(y) : y \in Y \} \)

• Form a weak dual pair if \( c(x) \leq w(y) \), \( \forall x \in X, \forall y \in Y \)

• Proposition. \( \max \{ c^T x : Ax \leq b, x \geq 0, \text{ integer} \} \) and
  \( \min \{ b^T y : A^T y \geq c, y \in \mathbb{R}_m \geq 0 \} \) form a weak dual pair.

• => any feasible solution to the dual of current (primal) LPR provides a valid upper bound.

• => means that don’t even need to solve dual to optimality. Can prune by bound earlier!

Branching Decision

• Most-fractional variable:
  – the variable with the fractional part closest to 0.5

• User priorities (e.g., “big decisions” first):
  – “the location of a facility is more consequential than the districts it serves” for example

• Strong branching:
  – “look ahead” before making a commitment to a branching decision

• Pseudocost method:
  – estimate effect on objective value of LPR of branching (approx form of strong branching,
    looks at what has happened earlier in problem)
Strong Branching: Example

Let $C$ denote set of integer vars in current subproblem with a fractional assignment

For each $j \in C$:
- (a) solve subproblem with $x_j \leq \lceil \bar{x}_j \rceil$ and $x_j \geq \lfloor \bar{x}_j \rfloor$
- (b) let $\bar{z}_j^D$ and $\bar{z}_j^U$ denote the value of LP solns

Branch on $j^* = \arg \min_{j \in C} \max[\bar{z}_j^D, \bar{z}_j^U]$

Can also solve subproblems approximately, just using a few dual simplex pivots.
Unboundedness in IPs

(advanced material)

• If an LP relaxation is unbounded:
  – if the integer variables take on integer values,
    then IP is unbounded. Else, we can branch.

• But, BnB may not terminate on a problem
  that is infeasible or unbounded.

• Will terminate if feasible, bounded, or if all
  integer variables bounded.

\[
\begin{align*}
\text{min } 0 \\
1 \leq 3x - 3y &\leq 2 \\
x, y &\in \mathbb{Z}
\end{align*}
\]

Infeasible, but B&B keeps searching

Formulation Strength
Recall: Firehouse Location

Formulation (P₁):
• If m=2 sites, and n=4 districts:
  • $x_{11} + x_{21} + x_{31} + x_{41} \leq 4y_1$
  • $x_{12} + x_{22} + x_{32} + x_{42} \leq 4y_2$
  • m constraints

Alternate formulation (P₂):
• $x_{11} \leq y_1; x_{21} \leq y_1; x_{31} \leq y_1; x_{41} \leq y_1$
• $x_{12} \leq y_2; x_{22} \leq y_2; x_{32} \leq y_2; x_{42} \leq y_2$
• mn constraints

• **Definition.** The **LP relaxation** (LPR) of an IP replaces all integer variables with continuous variables.
• **Definition.** The **polyhedron of an IP** is the feasible region of the LPR.
Valid formulations

• Consider \((\text{IP}) \max \{c^T x: x \in S \subseteq \mathbb{Z}^n\}\)

• **Defn.** Polyhedron \(P\) is a **valid formulation** for the \(\text{IP}\) if \(P \cap \mathbb{Z}^n = S\)

![Diagram showing points within valid formulations]

Strong Formulations

• **Proposition.** Consider two valid formulations \(P_1\) and \(P_2\), with \(P_2 \subseteq P_1 \subseteq \mathbb{R}^n\). Then \(z_{2\text{LP}} \leq z_{1\text{LP}}\).

• **Proof.** Suppose \(z_{2\text{LP}} > z_{1\text{LP}}\). But \(x^*\) that is best in \(P_2\) is also feasible in \(P_1\). Contradiction!

• Say that “\(P_2\) is **stronger** than \(P_1\).”
Example

Tight “big M”:
• \( x_{11} + x_{21} + x_{31} + x_{41} \leq 4y_1 \) \( P_1 \)

Loose “big M”:
• \( x_{11} + x_{21} + x_{31} + x_{41} \leq 10y_1 \) \( P_1' \)

• Both are valid formulations.
• But \( P_1' \) has additional fractional solutions, e.g. \( x=(0.5, 0.5, 0.5, 0.5), y=(0.25) \)
• \( \Rightarrow P_1 \) is stronger than \( P_1' \)

Importance of Strong Formulations

Benefit 1:
Improve LP bounding \( \rightarrow \) more fathoming of open problems by bound. (main advantage).

Benefit 2:
If search is best-bound first, better guidance in regard to node selection.

Benefit 3:
Fewer optimal, non-integral solutions.
**Convex Hull**

- **Definition.** Given set $X \subseteq \mathbb{Z}^n$, the **convex hull** of $X = \{x^1, \ldots, x^t\}$ is $\text{conv}(X) = \{x: x = \sum_{k=1}^{t} \lambda_k x^k, \sum_{k=1}^{t} \lambda_k = 1, \lambda_k \geq 0 \text{ for all } k\}$

- **Prop.** $\text{conv}(X)$ is a polyhedron.

- **Prop.** Extreme points of $\text{conv}(X)$ all lie in $X$.

- **Prop.** Can solve IP via solving LP on $\text{conv}(X)$.

**An “Ideal” formulation!**

Replace IP $\max\{c^T x: x \in X\}$ with the LP

$\{\max c^T x: x \in \text{conv}(X)\}$

- **Problem:** can require an exponential number of inequalities to define $\text{conv}(X)$
  - If $|X| = q$, may need as many as $2^q$ inequalities.
  - Better be the case, else we’d have P=NP!
Computational Complexity (Brief!)

- **Decision problems**: Is there a TSP tour cost \( \leq 10 \)?
- **P**: class of decision problems that can be solved in polynomial time (“easy”)
- **NP**: class of problems for which when answer is YES there is easy (poly time) proof (e.g., TSP)

- A problem is **NP-complete** if it is in NP and any problem in NP can be “reduced” (in poly time) to the problem; e.g. 0/1 integer programs.

**Widely conjectured that P \( \neq \) NP**

Complexity of LP

(advanced material)

- The simplex method is fast in practice, but not worst-case polynomial time.

- First **polynomial-time** LP algorithm was devised in 1979 by Khachian (made headlines!).
- Khachian’s **Ellipsoid method** is an interior point method. Does not rely on vertex solutions. Fits an increasingly good ellipsoid approximation. Poly time

- In 1984, Karmarkar announced a poly-time interior-point method with solution times 50x better than simplex. Again made headlines!
• LP is in class P, however 0/1 IP is NP-complete.
• If a polynomially-sized description of $\text{conv}(X)$ could be obtained for every IP, we could solve IPs in poly-time via reduction to LPs.

• Eureka! We would prove $P=NP$.

• So, some instances of an NP-hard problem (e.g., traveling salesperson) have a convex hull with an exp. num of extreme points; thus $2^q$ inequalities to describe…

Alternative Goal

• Q: What else can we do (other than formulate the convex hull) to improve the strength of formulations?

• A: Try to automatically approximate $\text{conv}(X)$ on a given instance, strengthening the formulation.
• Defn. A valid inequality may remove some fractional solns, but removes no integer solns.
• Defn. A cut is a valid inequality that removes the current fractional solution $x^\ast$.

Looking at CPLEX

• IBM’s Ilog CPLEX solver is used by AMPL
• Routinely used to solve real world problems of large economic significance
Some CPLEX features

• Automated Cut Generation
  – At “root” node (Global cuts)
  – At search nodes (Local cuts)

• Automated bound strengthening
  – Tighten right-hand side

• Primal heuristics
  – Look for integer feasible solutions that are close to current fractional solution

Example

```ampl
var X1 integer >= 0;
var X2 integer >= 0;
maximize Obj: 5 * X1 + 8 * X2;
subject to C1: X1 + X2 <= 6;
subject to C2: 5 * X1 + 9 * X2 <= 45;
end;
```

```ampl
ampl: model simple.mod;
ampl: option solver cplex;
ampl: solve;
```
Enabling feedback to AMPL

```ampl
option cplex_options
    'timing = 1'        # display timing info
    'mipdisplay=2'      # display MIP information
    'mipinterval=1';     # node interval

    timing=1: show how much cpu time used to solve the problem
    mipdisplay=2: show # of open nodes
    mipinterval=n: display information every n nodes and whenever it finds
                   an integer solution
```

Root relaxation solution time = 0.00 sec. (0.00 ticks)

<table>
<thead>
<tr>
<th>Nodes</th>
<th>Node</th>
<th>Left</th>
<th>Objective</th>
<th>IInf</th>
<th>Best Integer</th>
<th>Best Bound</th>
<th>Cuts</th>
<th>ItCnt</th>
<th>Gap</th>
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</table>

Elapsed time = 0.01 sec. (0.03 ticks, tree = 0.00 MB)

Root node processing (before b&c):
Real time = 0.01 sec. (0.03 ticks)
Parallel b&c, 16 threads:
Real time = 0.00 sec. (0.00 ticks)
Sync time (average) = 0.00 sec.
Wait time (average) = 0.00 sec.

Total (root+branch&cut) = 0.01 sec. (0.03 ticks)

Times (seconds):
Input = 0.000782
Solve = 0.018569
Output = 0.000492

CPLEX 12.6.0.0: optimal integer solution; objective 40
2 MIP simplex iterations
0 branch-and-bound nodes
Disabling Presolve

option presolve 0;
option cplex_options
  'timing = 1'
  'mipdisplay=2'
  'mipinterval=1'
  'boundstr=0'
  'dependency=0'
  'coeffreduce=0'
  'presolve=0'
  'cutpass= -1'
  'scale = -1'
  'prerelax = 0'
  'presolvenode = -1';

Disabling Primal Heuristics

option cplex_options
  'fpheur = -1'
  'heurfreq = -1'
  'rinsheur = -1';

---

fpheur: Whether to use the feasibility pump heuristic on MIP problems (find initial feasible):

-1 = no
0 = automatic choice (default)

heurfreq: How often to apply "node heuristics" for MIPS

-1= no
20=every twenty nodes

rinsheur: Relaxation INDuced neighborhood Search HEURistic for MIP problems:

-1 = none
0 = automatic choice of interval (default)
n (for n > 0) = every n nodes.
Root relaxation solution time = 0.00 sec. (0.00 ticks)

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Elapsed time = 0.01 sec. (0.05 ticks, tree = 0.01 MB)

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<tr>
<td>*</td>
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Root node processing (before b&c):
Real time = 0.00 sec. (0.01 ticks)
Parallel b&c, 16 threads:
Real time = 0.02 sec. (0.05 ticks)
Sync time (average) = 0.00 sec.
Wait time (average) = 0.00 sec.

Total (root+branch&cut) = 0.02 sec. (0.05 ticks)

Times (seconds):
Input = 0.000825
Solve = 0.091536
Output = 0.000714

CPLEX 12.6.0.0: optimal integer solution; objective 40
6 MIP simplex iterations
5 branch-and-bound nodes

Summary: Branch and Bound

- **Node Selection**
  - DFS together with dual pivots
  - Followed by best-first search (use LP bounds)

- **Branching decision**
  - Most-fractional, pseudo-cost based, user priorities, **strong branching**.

- **Formulation strength**
  - More fathoming by bound, better node selection guidance. Fewer fractional optimal solns

- **Cut generation**
  - Automatically tighten the formulation as search