

# AM 121: Intro to Optimization Models and Methods Fall 2018

## Lecture 10: Dual Simplex



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## Lesson Plan

- Interpret primal simplex in terms of pivots on the corresponding **dual tableau**
- **Dictionaries**
- Define **dual simplex** and interpret it in terms of pivots on the primal tableau
- **Applications**

Jensen & Bard: 3.9

## Motivation: Dual Simplex

- Useful for solving **integer programs** via branch-and-bound search.
- Find a solution after a new constraint is added, or an existing constraint is modified.
- A new “Phase 1 – Phase 2” method (no need for artificial variables!)

## Naming convention

$$\begin{aligned} \bullet \max & \sum_{j=m+1}^{m+n} c_j x_j \\ \text{s.t.} & \sum_{j=m+1}^{m+n} a_{ij} x_j \leq b_i \quad i = 1, 2, \dots, m \\ & x_j \geq 0 \quad j = m+1, \dots, m+n \end{aligned}$$

primal vars  $x_{m+1}, \dots, x_{m+n}$ ; slack vars  $x_1, \dots, x_m$

$$\begin{aligned} \bullet \min & \sum_{i=1}^m b_i y_i \\ \text{s.t.} & \sum_{i=1}^m a_{ij} y_i \geq c_j \quad j = m+1, \dots, m+n \\ & y_i \geq 0 \quad i = 1, \dots, m \end{aligned}$$

dual vars  $y_1, \dots, y_m$ ; surplus vars  $y_{m+1}, \dots, y_{m+n}$

## Example

$$\begin{aligned} \max \quad & 4x_3 + x_4 + 3x_5 \\ \text{s.t.} \quad & x_3 + 4x_4 \leq 1 \\ & 3x_3 - x_4 + x_5 \leq 3 \\ & x_3, x_4, x_5 \geq 0 \end{aligned}$$

Dual:

|                                  |                   |                                    |
|----------------------------------|-------------------|------------------------------------|
| $\min y_1 + 3y_2$                |                   | $\max -y_1 - 3y_2$                 |
| $\text{s.t. } y_1 + 3y_2 \geq 4$ |                   | $\text{s.t. } -y_1 - 3y_2 \leq -4$ |
| $4y_1 - y_2 \geq 1$              | $\longrightarrow$ | $-4y_1 + y_2 \leq -1$              |
| $y_2 \geq 3$                     | equivalent        | $-y_2 \leq -3$                     |
| $y_1, y_2 \geq 0$                | maximization      | $y_1, y_2 \geq 0$                  |
|                                  | problem           |                                    |

## Tableau (Review)

- A tableau must have isolated variables but **does not need to have** non-negative right hand side values.
- If the RHS is negative, the tableau is not (primal) feasible.

# Primal <> Dual

**Initial primal tableau:**

$$\begin{aligned} z & -4x_3 - x_4 - 3x_5 = 0 \\ x_1 & + x_3 + 4x_4 = 1 \\ x_2 & + 3x_3 - x_4 + x_5 = 3 \end{aligned}$$

- $B=\{1,2\}$   $B'=\{3,4,5\}$

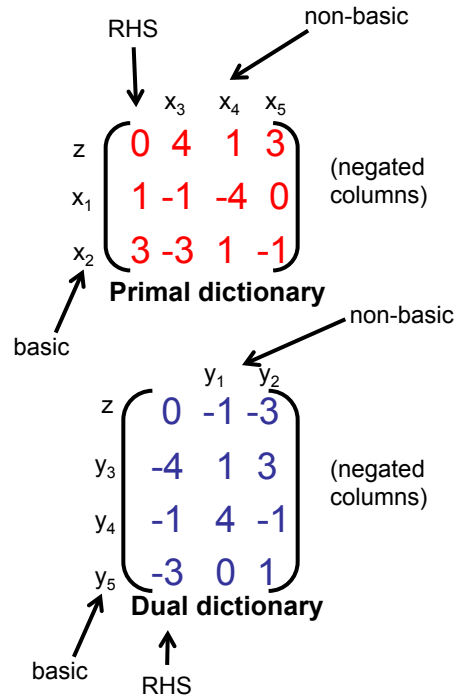
feasible

**Initial dual tableau:**

$$\begin{aligned} z + y_1 + 3y_2 & = 0 \\ -y_1 - 3y_2 + y_3 & = -4 \\ -4y_1 + y_2 + y_4 & = -1 \\ -y_2 + y_5 & = -3 \end{aligned}$$

- $B=\{3,4,5\}$   $B'=\{1,2\}$

infeasible



# Primal <> Dual

**Initial primal tableau:**

$$\begin{aligned} z & -4x_3 - x_4 - 3x_5 = 0 \\ x_1 & + x_3 + 4x_4 = 1 \\ x_2 & + 3x_3 - x_4 + x_5 = 3 \end{aligned}$$

- $B=\{1,2\}$   $B'=\{3,4,5\}$

Feasible; Neg shadow costs

$$\begin{pmatrix} 0 & 4 & 1 & 3 \\ 1 & -1 & -4 & 0 \\ 3 & -3 & 1 & -1 \end{pmatrix} B=\{1,2\}$$

Primal dictionary

**Initial dual tableau:**

$$\begin{aligned} z + y_1 + 3y_2 & = 0 \\ -y_1 - 3y_2 + y_3 & = -4 \\ -4y_1 + y_2 + y_4 & = -1 \\ -y_2 + y_5 & = -3 \end{aligned}$$

- $B=\{3,4,5\}$   $B'=\{1,2\}$

Infeasible; Non-neg shadow costs

$$\begin{pmatrix} 0 & -1 & -3 \\ -4 & 1 & 3 \\ -1 & 4 & -1 \\ -3 & 0 & 1 \end{pmatrix} B=\{3,4,5\}$$

Dual dictionary

-ve transpose of primal;  
B (dual) = B' (primal)

# A Dictionary

- **Definition.** The dictionary corresponding to a tableau with basis B is  $\begin{pmatrix} v & -\bar{c}_{B'}^T \\ \bar{b} & -\bar{A}_{B'} \end{pmatrix}$  (where B' is the set of non-basic) with rows of  $\bar{A}_{B'}$  and  $\bar{b}$  **ordered by indices** of basic vars and columns of  $\bar{c}_{B'}^T$  **ordered by indices** of non-basic vars

- Suppose tableau was reordered as:

$$\begin{aligned} z & & -x_4 - 4x_3 - 3x_5 & = 0 \\ & & x_2 - x_4 + 3x_3 + x_5 & = 3 \\ x_1 & & + 4x_4 + x_3 & = 1 \end{aligned}$$

Corresponding dictionary is still:

$$\begin{pmatrix} 0 & 4 & 1 & 3 \\ 1 & -1 & -4 & 0 \\ 3 & -3 & 1 & -1 \end{pmatrix} \quad B=\{1,2\}$$

- **A dictionary and basis B completely defines a tableau!**

## A simple primal-dual correspondence

negated transpose

Primal dictionary  $\leftrightarrow$  Dual dictionary

$$0 \ 4 \ 1 \ 3$$

$$1 \ -1 \ -4 \ 0$$

$$3 \ -3 \ 1 \ -1$$

Basic B, non-basic B'

$$0 \ -1 \ -3$$

$$-4 \ 1 \ 3$$

$$-1 \ 4 \ -1$$

$$-3 \ 0 \ 1$$

Basic B', non-basic B

- So: (1) primal tableau has corresponding dual tableau  
 (2) can track progress of simplex in the dual!  
 (3) primal obj value = - dual obj value at each step  
 (negated because we converted min f(y) into max -f(y))

## Primal pivot (track in dual)

$$\begin{array}{rcl}
 z & -4x_3 - x_4 - 3x_5 & = 0 \\
 x_1 & + x_3 + 4x_4 & = 1 \\
 x_2 & + 3x_3 - x_4 + x_5 & = 3 \\
 B & = \{1,2\} & B' = \{3,4,5\}
 \end{array}
 \quad
 \begin{array}{rcl}
 z & +3x_2 + 5x_3 - 4x_4 & = 9 \\
 x_1 & + x_3 + 4x_4 & = 1 \\
 x_2 & + 3x_3 - x_4 + x_5 & = 3 \\
 B & = \{1,5\} & B' = \{2,3,4\}
 \end{array}$$

pivot (2,5); 2 out 5 in  $\xrightarrow{\text{pivot}}$

$$\begin{array}{rcl}
 z + y_1 + 3y_2 & = & 0 \\
 -y_1 - 3y_2 + y_3 & = & -4 \\
 -4y_1 + y_2 + y_4 & = & -1 \\
 -y_2 + y_5 & = & -3 \\
 B & = \{3,4,5\} & B' = \{1,2\}
 \end{array}
 \quad
 \begin{array}{rcl}
 z + y_1 + 3y_5 & = & -9 \\
 -y_1 + y_3 - 3y_5 & = & 5 \\
 -4y_1 + y_4 + y_5 & = & -4 \\
 y_2 - y_5 & = & 3 \\
 B & = \{2,3,4\} & B' = \{1,5\}
 \end{array}$$

pivot (5,2); 5 out 2 in  $\xrightarrow{\text{pivot}}$  **still dual infeasible**

## New dictionaries

$$\begin{array}{rcl}
 z & +3x_2 + 5x_3 - 4x_4 & = 9 \\
 x_1 & + x_3 + 4x_4 & = 1 \\
 x_2 & + 3x_3 - x_4 + x_5 & = 3 \\
 B & = \{1,5\} & B' = \{2,3,4\}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{ccc}
 & x_2 & x_3 & x_4 \\
 z & \left( \begin{array}{ccc}
 9 & -3 & -5 & 4 \\
 1 & 0 & -1 & -4 \\
 3 & -1 & -3 & 1
 \end{array} \right) \\
 x_1 & & & \\
 x_5 & & & 
 \end{array} \\
 \text{Primal dictionary}
 \end{array}$$

$$\begin{array}{rcl}
 z + y_1 + 3y_5 & = & -9 \\
 -y_1 + y_3 - 3y_5 & = & 5 \\
 -4y_1 + y_4 + y_5 & = & -4 \\
 y_2 - y_5 & = & 3 \\
 B & = \{2,3,4\} & B' = \{1,5\}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{cc}
 & y_1 & y_5 \\
 z & \left( \begin{array}{cc}
 -9 & -1 & -3 \\
 3 & 0 & 1 \\
 5 & 1 & 3 \\
 -4 & 4 & -1
 \end{array} \right) \\
 y_2 & & \\
 y_3 & & \\
 y_4 & & 
 \end{array} \\
 \text{Dual dictionary} \\
 \text{(negated transpose)}
 \end{array}$$

## Primal pivot (track in dual)

$$\begin{array}{rcl}
 z + 3x_2 + 5x_3 - 4x_4 & = & 9 \\
 x_1 + x_3 + 4x_4 & = & 1 \\
 x_2 + 3x_3 - x_4 + x_5 & = & 3
 \end{array}
 \quad
 \begin{array}{rcl}
 z + x_1 + 3x_2 + 6x_3 & = & 10 \\
 \frac{1}{4}x_1 + \frac{1}{4}x_3 + x_4 & = & \frac{1}{4} \\
 \frac{1}{4}x_1 + x_2 + 3\frac{1}{4}x_3 + x_5 & = & 3\frac{1}{4}
 \end{array}$$

$B = \{1, 5\}$   $B' = \{2, 3, 4\}$ 

 $B = \{4, 5\}$   $B' = \{1, 2, 3\}$

Pivot (1,4); 1 out 4 in
pivot
Primal feasible,  
Primal optimal.

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$$\begin{array}{rcl}
 z + y_1 + 3y_5 & = & -9 \\
 -y_1 + y_3 - 3y_5 & = & 5 \\
 -4y_1 + y_4 + y_5 & = & -4 \\
 y_2 - y_5 & = & 3
 \end{array}
 \quad
 \begin{array}{rcl}
 z + \frac{1}{4}y_4 + 3\frac{1}{4}y_5 & = & -10 \\
 y_1 - \frac{1}{4}y_4 - \frac{1}{4}y_5 & = & 1 \\
 y_2 - y_5 & = & 3 \\
 y_3 - \frac{1}{4}y_4 - 3\frac{1}{4}y_5 & = & 6
 \end{array}$$

$B = \{2, 3, 4\}$   $B' = \{1, 5\}$ 

 $B = \{1, 2, 3\}$   $B' = \{4, 5\}$

Pivot (4,1); 4 out 1 in
pivot
Dual feasible,  
Dual optimal

## Final tableaus; Final dictionaries

$$\begin{array}{rcl}
 z + x_1 + 3x_2 + 6x_3 & = & 10 \\
 \frac{1}{4}x_1 + \frac{1}{4}x_3 + x_4 & = & \frac{1}{4} \\
 \frac{1}{4}x_1 + x_2 + 3\frac{1}{4}x_3 + x_5 & = & 3\frac{1}{4}
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{cccc}
 & x_1 & x_2 & x_3 \\
 z & 10 & -1 & -3 & -6 \\
 x_4 & \frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} \\
 x_5 & 3\frac{1}{4} & -\frac{1}{4} & -1 & -3\frac{1}{4}
 \end{array} \\
 \text{Primal dictionary}
 \end{array}$$

$B = \{4, 5\}$   $B' = \{1, 2, 3\}$

$$\begin{array}{rcl}
 z + \frac{1}{4}y_4 + 3\frac{1}{4}y_5 & = & -10 \\
 y_1 - \frac{1}{4}y_4 - \frac{1}{4}y_5 & = & 1 \\
 y_2 - y_5 & = & 3 \\
 y_3 - \frac{1}{4}y_4 - 3\frac{1}{4}y_5 & = & 6
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{ccc}
 & y_4 & y_5 \\
 z & -10 & -\frac{1}{4} & -3\frac{1}{4} \\
 y_1 & 1 & \frac{1}{4} & \frac{1}{4} \\
 y_2 & 3 & 0 & 1 \\
 y_3 & 6 & \frac{1}{4} & 3\frac{1}{4}
 \end{array} \\
 \text{Dual dictionary} \\
 \text{(negated transpose)}
 \end{array}$$

$B = \{1, 2, 3\}$   $B' = \{4, 5\}$

## From Initial to Final tableau

| Initial primal tableau                | Final primal tableau  |
|---------------------------------------|---|
| $z - 4x_3 - x_4 - 3x_5 = 0$           | $z + x_1 + 3x_2 + 6x_3 = 10$                                  |
| $x_1 + x_3 + 4x_4 = 1$                | $\frac{1}{4}x_1 + \frac{1}{4}x_3 + x_4 = \frac{1}{4}$         |
| $x_2 + 3x_3 - x_4 + x_5 = 3$          | $\frac{1}{4}x_1 + x_2 + 3\frac{1}{4}x_3 + x_5 = 3\frac{1}{4}$ |
| $B = \{1, 2\} \quad B' = \{3, 4, 5\}$ | $B = \{4, 5\} \quad B' = \{1, 2, 3\}$                         |

| Initial dual tableau                  | Final dual tableau                           |
|---------------------------------------|--|
| $z + y_1 + 3y_2 = 0$                  | $z + \frac{1}{4}y_4 + 3\frac{1}{4}y_5 = -10$ |
| $-y_1 - 3y_2 + y_3 = -4$              | $y_1 - \frac{1}{4}y_4 - \frac{1}{4}y_5 = 1$  |
| $-4y_1 + y_2 + y_4 = -1$              | $y_2 - y_5 = 3$                              |
| $-y_2 + y_5 = -3$                     | $y_3 - \frac{1}{4}y_4 - 3\frac{1}{4}y_5 = 6$ |
| $B = \{3, 4, 5\} \quad B' = \{1, 2\}$ | $B = \{1, 2, 3\} \quad B' = \{4, 5\}$        |

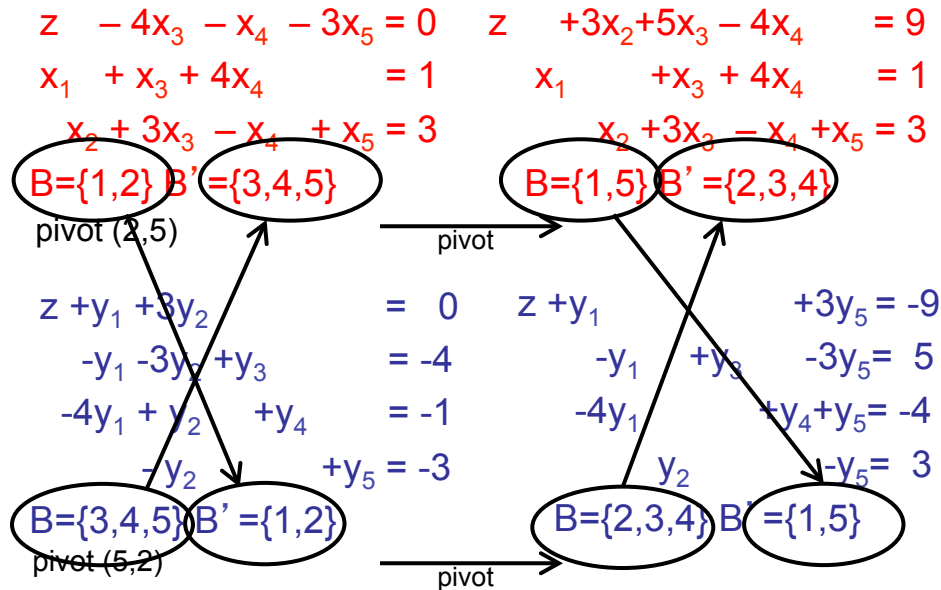
## Primal Simplex: Effect On Dual Solution

- Maintains primal feasibility, terminates with primal optimality
  - Non-negative RHS in primal  $\leftrightarrow$  non-negative reduced costs in dual
  - Free reduced costs in primal  $\leftrightarrow$  free RHS values in dual (and dual infeasible for a while)
  - Terminates with non-negative reduced costs in primal  $\leftrightarrow$  non-neg RHS in dual (and dual feasible)
  - “Pivot in primal, track in dual.”
- Corresponding dual solution always has non-negative reduced costs, but is initially infeasible.

Note: pair (x,y) satisfy complementary slackness during pivots, but y is infeasible until termination.



## Complementary Slackness



It is because of CS that we get this particular dual  $\leftrightarrow$  primal construction

Primal dictionary  $\leftrightarrow$  Dual dictionary

|                   |                   |               |
|-------------------|-------------------|---------------|
| $0 \ 4 \ 1 \ 3$   | negated transpose | $0 \ -1 \ -3$ |
| $1 \ -1 \ -4 \ 0$ |                   | $-4 \ 1 \ 3$  |
| $3 \ -3 \ 1 \ -1$ |                   | $-1 \ 4 \ -1$ |
|                   |                   | $-3 \ 0 \ 1$  |

Basic  $B$ , non-basic  $B'$

Basic  $B'$ , non-basic  $B$

In particular, it is CS that gives us that:

primal obj value = - dual obj value at each step

(negated because we converted  $\min f(y)$  into  $\max -f(y)$ )

## Different kinds of pivots

- **V1**: pick non-basic  $x_j$  with  $\bar{c}_j < 0$  to enter, and look at positive column entries to find basic var to leave
- **V2**: pick basic  $x_j$  with  $\bar{b}_j < 0$  to exit, and look at negative row entries to find non-basic var to enter

|    | Primal simplex        | Dual simplex          |
|----|-----------------------|-----------------------|
| V1 | <b>Primal tableau</b> | <b>Dual tableau</b>   |
| V2 | <b>Dual tableau</b>   | <b>Primal tableau</b> |

## Dual Simplex

- Maintain dual feasibility, terminate with dual optimality
- Rather than work in dual tableau, can track in primal tableau: perform "**dual pivots**".

# Dual Simplex

- Maintain dual feasibility, terminate with dual optimality
- Rather than work in dual tableau, can track in primal tableau: perform "**dual pivots**".

Primal tableau properties

|               | Primal feasible        | Dual feasible        | Primal and Dual optimal |
|---------------|------------------------|----------------------|-------------------------|
| Reduced costs | Free                   | Non-neg              | Non-neg                 |
| RHS           | Non-neg                | Free                 | Non-neg                 |
|               | primal pivots to solve | dual pivots to solve |                         |

## Example: Dual Simplex

Primal problem

$$\begin{aligned}
 &\max -x_4 \quad -x_5 \\
 &\text{s.t. } -2x_4 \quad -x_5 \leq 4 \\
 &\quad -2x_4 + 4x_5 \leq -8 \\
 &\quad -x_4 + 3x_5 \leq -7 \\
 &\quad x_4, \quad x_5 \geq 0
 \end{aligned}$$

Dual problem

$$\begin{aligned}
 &\min 4y_1 - 8y_2 - 7y_3 && \max -4y_1 + 8y_2 + 7y_3 \\
 &\text{s.t. } -2y_1 - 2y_2 - y_3 \geq -1 && \text{s.t. } 2y_1 + 2y_2 + y_3 \leq 1 \\
 &\quad -y_1 + 4y_2 + 3y_3 \geq -1 && \quad y_1 - 4y_2 - 3y_3 \leq 1 \\
 &\quad y_1, y_2, y_3 \geq 0 && \quad y_1, y_2, y_3 \geq 0
 \end{aligned}$$

$\xrightarrow{\text{equivalent maximization problem}}$

## Initial Tableau

$$\begin{array}{rcl}
 \max & -x_4 & -x_5 & z & +x_4 + x_5 & = & 0 \\
 \text{s.t.} & -2x_4 & -x_5 & x_1 & -2x_4 - x_5 & = & 4 \\
 & -2x_4 & +4x_5 & x_2 & -2x_4 + 4x_5 & = & -8 \\
 & -x_4 & +3x_5 & x_3 & -x_4 + 3x_5 & = & -7 \\
 & x_4, & x_5 & & & \geq & 0
 \end{array}$$

$$\begin{array}{rcl}
 \max & -4y_1 & +8y_2 & +7y_3 & z & +4y_1 -8y_2 -7y_3 & = & 0 \\
 \text{s.t.} & 2y_1 & +2y_2 & +y_3 & & 2y_1 + 2y_2 + y_3 + y_4 & = & 1 \\
 & y_1 & -4y_2 & -3y_3 & & y_1 - 4y_2 - 3y_3 + y_5 & = & 1 \\
 & y_1, & y_2, & y_3 & & \geq & 0
 \end{array}$$

## Dual Pivot (track in primal)

$$\begin{array}{rcl}
 z & +x_4 & +x_5 & = & 0 & z & +\frac{1}{2}x_2 & +3x_5 & = & -4 \\
 x_1 & -2x_4 & -x_5 & = & 4 & x_1 & -x_2 & -5x_5 & = & 12 \\
 x_2 & -2x_4 & +4x_5 & = & -8 & & -\frac{1}{2}x_2 & +x_4 - 2x_5 & = & 4 \\
 & x_3 - x_4 & +3x_5 & = & -7 & & -\frac{1}{2}x_2 + x_3 & +x_5 & = & -3
 \end{array}$$

$$B = \{1, 2, 3\}$$

pivot (2,4) (2 out, 4 in)

→ pivot

$$B = \{1, 3, 4\}$$

$$\begin{array}{rcl}
 z & +4y_1 & -8y_2 & -7y_3 & = & 0 & z & +12y_1 & -3y_3 & +4y_4 & = & +4 \\
 & 2y_1 & +2y_2 & +y_3 & +y_4 & = & 1 & & y_1 + y_2 + \frac{1}{2}y_3 + \frac{1}{2}y_4 & = & \frac{1}{2} \\
 & y_1 & -4y_2 & -3y_3 & +y_5 & = & 1 & & 5y_1 & -y_3 & +2y_4 & +y_5 & = & 3
 \end{array}$$

$$B = \{4, 5\}$$

↑ pivot (4,2) (4 out, 2 in)

$$B = \{2, 5\}$$

## Dual Pivot (track in primal)

|   |                            |  |
|---|----------------------------|--|
| $  \begin{array}{rcl}  z & +\frac{1}{2}x_2 & +3x_5 = -4 \\  x_1 & -x_2 & -5x_5 = 12 \\  & -\frac{1}{2}x_2 & +x_4 - 2x_5 = 4 \\  & \boxed{-\frac{1}{2}x_2} & +x_3 + x_5 = -3 \\  B & = \{1, 3, 4\} & \\  & \text{pivot } (3, 2) &   \end{array}  $ | $\longrightarrow$<br>pivot | $  \begin{array}{rcl}  z & +x_3 & +4x_5 = -7 \\  x_1 & -2x_3 & -7x_5 = 18 \\  & -x_3 + x_4 & -3x_5 = 7 \\  & x_2 - 2x_3 & -2x_5 = 6 \\  B & = \{1, 2, 4\} &   \end{array}  $ |
| $  \begin{array}{rcl}  z & +12y_1 & -3y_3 + 4y_4 = +4 \\  & y_1 + y_2 + \boxed{\frac{1}{2}y_3} & +\frac{1}{2}y_4 = \frac{1}{2} \\  & 5y_1 & -y_3 + 2y_4 + y_5 = 3 \\  B & = \{2, 5\} & \\  & \text{pivot } (2, 3) &   \end{array}  $              |                            | $  \begin{array}{rcl}  z & +18y_1 + 6y_2 & +7y_4 = +7 \\  & 2y_1 + 2y_2 + y_3 + y_4 & = 1 \\  & 7y_1 + 2y_2 + 3y_4 + y_5 & = 4 \\  B & = \{3, 5\} &   \end{array}  $         |

OPTIMAL!

## “Upside Down Pivoting”

- Dual pivots in primal tableau. Maintain dual feasibility, terminate with dual optimality (non-negative RHS values in primal tableau).
- Primal infeasible for a while, but non-negative (primal) reduced costs. Terminate with feasible primal solution.
- 
- Choose a variable with a **-ve RHS to exit**, and do ratio test on **strictly negative entries in row** with **negated reduced cost as numerator**.

## Dual simplex

- Assume have a dual feasible tableau ( $\bar{c}_B \geq 0$ )
  - **Step 1.** Pick a basic variable  $x_r$  to **leave** with a *strictly negative RHS*  $\bar{b}_r < 0$ . If no such variable then optimal and **STOP**.
  - **Step 2.** Pick a nonbasic variable  $x_k$  to **enter** by considering row  $r$  and non-basic variables with **strictly negative** coefficients. *Ratio test:*  $k$  should satisfy  $-\bar{c}_k / \bar{a}_{rk} = \min \{-\bar{c}_j / \bar{a}_{rj} : j \notin B, \bar{a}_{rj} < 0\}$ .  
If all  $\bar{a}_{rj}$  on  $j \notin B$  nonnegative, then dual unbounded and primal infeasible and **STOP**.
  - **Step 3.** *Pivot* on  $(r,k)$  and go to step 1.
- 
- In corresponding dual tableau: basic variable  $y_r$  enters and nonbasic variable  $y_k$  leaves.

## Example: A dual simplex pivot

$$\begin{array}{rcl}
 Z & & +\frac{1}{2}x_3 + \frac{1}{2}x_4 = +1\frac{1}{2} \\
 x_1 & & +x_3 = 2 \\
 x_2 & \boxed{-\frac{1}{2}x_3} & +\frac{1}{2}x_4 = -\frac{1}{2}
 \end{array}$$

$$\begin{array}{rcl}
 Z & +x_2 & +x_4 = 1 \\
 x_1 + 2x_2 & & +x_4 = 1 \\
 -2x_2 + x_3 - x_4 & & = 1
 \end{array}$$

(Pivot on row with -ve RHS, picking nonbasic variable to enter with strictly negative  $\bar{a}_{rk}$  and minimal ratio  $-\bar{c}_k / \bar{a}_{rk}$ )

OPTIMAL!

## Example: Dual unbounded

- Suppose get to a (primal) tableau with row
$$x_1 + 2x_2 + 2x_3 = -5$$
- Dual unbounded, since coefficients on  $x_2$  and  $x_3$  are non-negative.
- Conclude that primal is infeasible (from weak duality theorem).

## Making use of the Dual Simplex

1. Find a solution after a new constraint is added (useful for sensitivity analysis, and solving IPs via branch-and-bound search)
2. Find a new solution after a change in a RHS coeff that is outside the allowable range
3. A new “Phase 1 – Phase 2” method (no need for artificial variables!)

# 1. Adding a New Constraint

- Recall the furniture example
- $$\begin{aligned} \max z &= 60x_1 + 30x_2 + 20x_3 \\ \text{s.t. } 8x_1 + 6x_2 + x_3 + x_4 &= 48 \quad (\text{lumber}) \\ 4x_1 + 2x_2 + 1.5x_3 + x_5 &= 20 \quad (\text{finishing}) \\ 2x_1 + 1.5x_2 + 0.5x_3 + x_6 &= 8 \quad (\text{carpentry}) \\ x_1, \dots, x_6 &\geq 0 \end{aligned}$$

$x_1$  desks;  $x_2$  tables;  $x_3$  chairs
- $B=\{4,3,1\}$ .  $x^*=(2,0,8,24,0,0)$ ,  $z=280$
- Three kinds of new constraints:
  - (i) Current optimal solution still feasible; e.g., add constraint  $x_1 + x_2 + x_3 \leq 11$ . Easy: just check.
  - (ii) Current basis becomes infeasible
  - (iii) LP becomes infeasible

## 1a. New constraint makes optimal basis infeasible.

- Add:  $x_2 \geq 1$ . Can just add directly to optimal tableau:
- $$\begin{aligned} z &+ 5x_2 + 10x_5 + 10x_6 = 280 \\ -2x_2 + x_4 + 2x_5 - 8x_6 &= 24 \\ -2x_2 + x_3 + 2x_5 - 4x_6 &= 8 \\ x_1 + 1.25x_2 - 0.5x_5 + 1.5x_6 &= 2 \\ -x_2 &+ x_7 = -1 \end{aligned}$$
- (primal)  $B=\{1,3,4,7\}$ . Dual feasible.
- Dual pivot in primal (7 out, 2 in). And so pivot (7,2):
- $$\begin{aligned} z &+ 10x_4 + 10x_5 + 5x_7 = 275 \\ x_4 + 2x_5 - 8x_6 - 2x_7 &= 26 \\ x_3 + 2x_5 - 4x_6 - 2x_7 &= 10 \\ x_1 - \frac{1}{2}x_5 + \frac{1}{2}x_6 + \frac{1}{4}x_7 &= \frac{3}{4} \\ x_2 - x_7 &= 1 \end{aligned}$$
- Find an optimal solution:  $x^*=(\frac{3}{4}, 1, 10, 26, 0, 0)$ ;  $z=275$ .



## Remark

- Could add  $x_2 \geq 1$  directly to the initial tableau:  

$$x_2 - x_7 + x_8 = 1,$$
 with artificial variable  $x_8$ , and re-solve with primal simplex.
- By adding the constraint to the final tableau and using dual simplex we just need a single pivot.
- Large computational gain!

### 1b. New constraint makes problem infeasible.

- Consider:  $x_1 + x_2 \geq 12$ 

$$\begin{array}{rccccrcr} z & & +5x_2 & & + 10x_5 & +10x_6 & = 280 \\ & & -2x_2 & + x_4 & + 2x_5 & - 8x_6 & = 24 \\ & & -2x_2 + x_3 & + & 2x_5 & - 4x_6 & = 8 \\ & x_1 + 1.25x_2 & & & -0.5x_5 + 1.5x_6 & & = 2 & (4) \\ & -x_1 & - x_2 & & & & + x_7 = -12 & (5) \end{array}$$
- $x_1$  no longer isolated. Adopt:  $(5') := (4) + (5)$ 

$$0.25x_2 - 0.5x_5 + 1.5x_6 + x_7 = -10 \quad (5')$$
- (Primal)  $B = \{4, 3, 1, 7\}$ . Dual feasible.
- Dual pivot in primal: (7 out, 5 in).
- Get tableau on next slide.

1b. New constraint makes problem infeasible.

$$\begin{array}{rcl}
 z & +10x_2 & +40x_6 +20x_7 = 80 \\
 & -x_2 & +x_4 -2x_6 + 4x_7 = -16 \\
 \rightarrow & \boxed{-x_2} + x_3 + & 2x_6 + 4x_7 = -32 \\
 & x_1 + x_2 & -x_7 = 12 \\
 & -\frac{1}{2}x_2 + & x_5 - 3x_6 - 2x_7 = 12 \\
 & -\frac{1}{2}x_2 + & x_5 - 3x_6 - 2x_7 = 20
 \end{array}$$

- (Primal)  $B=\{4,3,1,5\}$
- Dual pivot in primal: 3 out, 2 in
- Get tableau on next slide

1b. New constraint makes problem infeasible.

$$\begin{array}{rcl}
 z & +10x_3 & +60x_6 +60x_7 = -240 \\
 & -x_3 +x_4 & -4x_6 = -16 \\
 & x_2 - x_3 & -2x_6 -4x_7 = 32 \\
 \boxed{x_1} & +x_3 & +2x_6 +3x_7 = -20 \\
 & -\frac{1}{2}x_3 & +x_5 - 4x_6 -4x_7 = 36
 \end{array}$$

- (Primal)  $B=\{4,2,1,5\}$
- Dual pivot in primal: “1 out.” But ratio test fails, no variable to enter.
- Dual unbounded, and primal infeasible (by weak duality).

## Making use of the Dual Simplex

1. Find a solution after a new constraint is added (useful for sensitivity analysis, and solving IPs via branch-and-bound search)
2. Find a new solution after a change in a RHS coeff that is outside the allowable range
3. A new “Phase 1 – Phase 2” method (no need for artificial variables!)

## 2. Changing a RHS

- Recall BFS for  $B=\{1,3,4\}$  in furniture problem remains feasible for  $16 \leq b_2 \leq 24$ . Suppose  $b_2:=30$ .
- Let's consider what happens at same basis.

- $z=y^Tb=(0 \ 10 \ 10) \begin{pmatrix} 48 \\ 30 \\ 8 \end{pmatrix} = 380$

$$\bar{b} = A_B^{-1}b = \begin{pmatrix} 1 & 2 & -8 \\ 0 & 2 & -4 \\ 0 & -0.5 & 1.5 \end{pmatrix} \begin{pmatrix} 48 \\ 30 \\ 8 \end{pmatrix} = \begin{pmatrix} 44 \\ 28 \\ -3 \end{pmatrix}$$

- Construct tableau. Next slide.

$$\begin{aligned}
 \bullet \quad z &+ 5x_2 && + 10x_5 + 10x_6 = 380 \\
 &- 2x_2 &+ x_4 + 2x_5 &- 8x_6 = 44 \\
 &- 2x_2 + x_3 + && 2x_5 - 4x_6 = 28 \\
 &x_1 + 1.25x_2 && \boxed{-0.5x_5} + 1.5x_6 = -3
 \end{aligned}$$

$B=\{1,3,4\}$ . Primal infeasible basis. Do a dual pivot in primal (1 out, 5 in.)

$$\begin{aligned}
 z &+ 20x_1 + 30x_2 && + 40x_6 = 320 \\
 &4x_1 + 3x_2 &+ x_4 &- 2x_6 = 32 \\
 &4x_1 + 3x_2 + x_3 && + 2x_6 = 16 \\
 &- 2x_1 - 2\frac{1}{2}x_2 &+ x_5 &- 3x_6 = 6
 \end{aligned}$$

$B=\{3,4,5\}$

OPTIMAL! Solution  $x^*=(0,0,16,32,6,0)$ . Only make chairs!

## Making use of the Dual Simplex

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2. Find a new solution after a change in a RHS coeff that is outside the allowable range
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### 3. New “Phase 1 - Phase 2” method

- (a)  $\bar{b} \geq 0$ : use **primal simplex** (no phase 1)
- (b)  $\bar{c} \geq 0$ , some  $\bar{b}_i < 0$ : use **dual simplex** (no phase 1)
- (c) Some  $\bar{b}_i < 0$  and some  $\bar{c}_j < 0$ 
  - E.g., suppose “max  $x_1 - 2x_2$ .”
  - First solve “max  $-x_1 - 2x_2$ .” This has  $z + x_1 + 2x_2$  first row, and so dual feasible. **Use dual simplex** to find a primal feasible solution.
  - Second, modify tableau to introduce correct objective. **Use primal simplex.**
  
  - ALTERNATIVE: Replace RHS  $\bar{b}$  with  $\bar{b} \geq 0$ , use primal simplex to find a dual feasible solution, then bring back correct RHS and use dual simplex.

### Summary: Dual Simplex

- **Dual simplex**: simplex method applied to the dual problem. Maintains a **dual feasible basis**, **terminates with an optimal dual basis**.
- We **track the dual simplex on primal tableau**. “Upside down pivots”: first which basic var exits (-ve RHS), then which non-basic enters?
- Flexible approach: can work in primal tableau and move seamlessly from primal to dual pivots.
- Useful when constraints change, for phase 1-phase 2, and we’ll use for solving IPs!