Problem Set 4
Duality, Sensitivity, Dual Simplex, Complementary Slackness

AM121/ES121 — Fall 2015

Due 5:00 PM, Friday, October 14, 2016 [**No late days**]

Announcements

- The assignment is due by 5:00 PM, Friday, October 14, 2016 [**No late days**]. **This is a hard deadline so you can study for the midterm on 10/17 and we can distribute a solution. Absolutely no late days are allowed on this assignment.**
- Readings: Jensen and Bard, 3.7-4.

Goals

This assignment has a number of goals. First, you will get practice converting a primal problem to its dual form. Second, you will get practice performing sensitivity analysis. Then, you will apply your knowledge of duality and sensitivity by serving the public sector. Finally, you will practice with complementary slackness and the dual simplex method by helping out a local farmer.

Contents

1 Make it dual

Consider the following linear program:

$$\begin{array}{rll}
\text{maximize} & 2x_1 + 4x_2 + 2x_3 + 2x_4 - 3x_5 - 2x_6 & \\
\text{subject to:} & 3x_1 + 5x_2 + 2x_4 + 3x_6 & \leq 29 \\
& 4x_1 + 2x_2 + 4x_4 - 3x_5 + 2x_6 & = \ 10 \\
& x_1 + 5x_3 - 5x_5 & \leq 20 \\
& -3x_2 + 2x_4 + 2x_5 & \leq 13 \\
& x_1 + 3x_2 + 2x_3 - x_4 + x_5 + 3x_6 & \leq 35 \\
& x_1, x_3, x_5, x_6 \geq 0; x_2, x_4 \ free &
\end{array}$$

Task 1

1. Solve this linear program in AMPL. Record the input file (no data file necessary) along with the feasible solution and objective value.
2. Write down the dual linear program in AMPL. Record the input file (no data file necessary) along with the feasible solution and objective value.
3. Ms. Trelawney is convinced that AMPL is cursed and trusting AMPL's solution is unwise. Verify by hand calculation that AMPL's solution to the primal problem is indeed optimal. Justify your answer by leveraging concepts from duality (without using the Simplex method).

End Task 1

2 Be sensitive

Consider the following linear program in standard equality form:

$$\begin{array}{rl}
\text{maximize} & c^T x \\
\text{subject to} & Ax = b \\
& x \geq 0
\end{array}$$

where

$$A = \begin{bmatrix} 5 & -9 & 9 & 1 & 0 & 0 \\ -8 & -2 & -7 & 0 & 1 & 0 \\ -4 & 7 & 10 & 0 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix}, c = \begin{bmatrix} -6 \\ 3 \\ 7 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Task 2

For this exercise, feel free to make use of Mathematica or Matlab for your calculations.

1. Consider the basis $B = [1, 2, 5]$. Verify that the inverse of the basis is $A_B^{-1} = \begin{bmatrix} -7 & 0 & -9 \\ -4 & 0 & -5 \\ -64 & 1 & -82 \end{bmatrix}$.
2. Calculate directly (e.g. without pivoting) the tableau with respect to B.

3. Verify that the corresponding basic solution is optimal.
4. Verify that the solution is nondegenerate.
5. By reasoning about the shadow prices (dual values) on the constraints, if b_2 changes by some small number ϵ, what happens to the optimal value?
6. By reasoning about the shadow prices (dual values) on theconstraints, if b_3 changes by some small number ϵ, what happens to the optimal value?
7. By reasoning about the effect of a change in the RHS on the final tableau, for what range of change ϵ to b_3 does the basic solution remain optimal? Show your work. (Hint: You can check your answer using the sensitivity ranges provided by CPLEX.)

End Task 2

3 Moo.

In your efforts to support local farmers, you decide to offer consulting services pro bono publico to *Mike's Milkin' Cattle Ranch.*

"*Everything we have here comes from our great Massachusetts cows. We produce three things: milk, cheese, and yogurt, which we sell for* \$2.50 *a pound,* \$2 *a pound, and* \$3 *a pound to the market. We get 120 pounds of raw material out of the cows, producing milk and cheese at 1:1 ratios and 2:1 for yogurt. The cultures eat half of the raw materials we throw in there. Anyhow, our buyer doesn't want us to produce more than 50 pounds of any particular good, so we keep that in mind. But other than that, we just want to produce 40 pounds of milk, 30 pounds of cheese and 25 pounds of yogurt. If you have a better plan, let us know.*"

You let Mike know that you will take care of it. You notice that the problem can be formulated as a linear program. Letting the decision variables x_1, x_2, and x_3 represent the number of pounds of milk, cheese, and yogurt to be produced, you formulate the following linear program in standard inequality form:

$$\begin{aligned} \text{maximize} \quad & 2.5x_1 + 2x_2 + 3x_3 \\ \text{subject to:} \quad & x_1 + x_2 + 2x_3 \leq 120 \\ & x_1 \leq 50 \\ & x_2 \leq 50 \\ & x_3 \leq 50 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

where you are maximizing profit subject to the production constraint and the demand constraints. Before solving the linear program, you decide to check the optimality of his current production levels.

Task 3

1. Before solving the linear program, check whether Mike's solution is optimal. (Hint: you will have to formulate the dual problem and then look to see whether there is a solution satisfying the complementary slackness condition with the primal solution.)
2. Solve the linear program (by inspection is fine). Confirm the optimality of your solution using complementary slackness; i.e., by working the dual formulation, imposing the requirements of complementary slackness with respect to your primal solution, and finding a feasible dual solution that satisfies these requirements.
3. What's the gain in profit over Mike's original production levels?

End Task 3

From Exercise Task 2, you remember that you can do a small bit of work to find the optimal tableau that corresponds to the optimal solution you had just found. First you write down the original LP in standard equality form with slack variables:

$$\begin{aligned} \text{maximize} \quad & 2.5x_1 + 2x_2 + 3x_3 \\ \text{subject to:} \quad & x_1 + x_2 + 2x_3 + s_0 = 120 \\ & x_1 + s_1 = 50 \\ & x_2 + s_2 = 50 \\ & x_3 + s_3 = 50 \\ & x_1, x_2, x_3, s_0, s_1, s_2, s_3 \geq 0 \end{aligned}$$

You deduce that the optimal tableau is as follows

$$\begin{bmatrix} & z & x_1 & x_2 & x_3 & s_0 & s_1 & s_2 & s_3 & rhs \\ z & 1 & 0 & 0 & 0 & 1.5 & 1 & 0.5 & 0 & 255 \\ x_1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 50 \\ x_2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 50 \\ x_3 & 0 & 0 & 0 & 1 & 0.5 & -0.5 & -0.5 & 0 & 10 \\ s_3 & 0 & 0 & 0 & 0 & -0.5 & 0.5 & 0.5 & 1 & 40 \end{bmatrix}$$

You decide to keep a copy of this, in case you need it for sensitivity analysis or other changes. While you are finishing up, Mike thanks you for your hard work. However, he had just heard word from the buyer that the ranch must produce more processed foods, such that the total production of yogurt and cheese is at least 70 pounds. "*What do we do?*"

Task 4

1. Write down this new constraint. Is the current optimal solution still feasible?
2. Instead of re-solving the problem from scratch, add this constraint (as an equality constraint with its associating slack variable) to the optimal tableau. Bring the tableau into dual-feasible form, and perform one or more dual-simplex pivots to solve the problem with the new constraint. What's the optimal solution and the corresponding profit?

End Task 4

4 Public Water System

Massachusetts's Public Water Distribution Department is looking for ways to reduce the operational costs associated with bringing water from the state reservoirs to local community water mains. Every week, the state transports enough water to each community water main to meet its estimated demand (there is always enough supply in the reservoir to meet demand). The state has recorded the estimated reservoir supply and community water demands (in million gallons):

	reservoir 1	reservoir 2	reservoir 3
supply	1700	2500	2600

	Cambridge	Boston	Watertown	Everett	Malden	Arlington	Medford
demand	900	1200	600	400	1500	1000	1000

The department has provided this data, along with the transportation costs per million gallons (in hundreds of dollars) from each reservoir to each local community, to two independent contractors to formulate models to minimize total transportation costs. The contractors have sent their results to you for review. You decide to start by looking at the second contractor's report.

Task 5

1. Read `transp2.mod` and `water2.dat`. Add comments to each line in `transp2.mod`. Make sure your comments explain the constraints in `transp2.mod`.
2. Use AMPL to solve the model, using CPLEX as your solver. What is the corresponding objective value and optimal solution? Be sure to include corresponding units.
3. Suppose the supply at reservoir 1 is now changed. What (if any) changes do you have to make to `transp2.mod` and `water2.dat`?

End Task 5

As the provided data are all subject to change, the department is interested in knowing how total transportation cost may be affected by changes in these numbers. The second contractor has provided his respective sensitivity analysis (resulting from running `transp2.run`). You can run this script in AMPL by typing `include transp2.run`:

```
Sensitivity Report from transp2.run (water2.sens):
:              _varname            _var.rc _var.down _var.current _var.up    :=
1    "Trans['RESV1','Cambridge']"    27         12          39       1e+20
2    "Trans['RESV1','Boston']"       14          0          14       1e+20
3    "Trans['RESV1','Watertown']"    11          0          11       1e+20
4    "Trans['RESV1','Everett']"      15         -1          14       1e+20
5    "Trans['RESV1','Malden']"        0         16          16          17
6    "Trans['RESV1','Arlington']"    56         26          82       1e+20
7    "Trans['RESV1','Medford']"       0          7           8           8
8    "Trans['RESV1','remainder']"    12        -12           0       1e+20
9    "Trans['RESV2','Cambridge']"     6         21          27       1e+20
10   "Trans['RESV2','Boston']"        0     -1e+20           9          11
11   "Trans['RESV2','Watertown']"     3          9          12       1e+20
12   "Trans['RESV2','Everett']"       1          8           9       1e+20
13   "Trans['RESV2','Malden']"        1         25          26       1e+20
14   "Trans['RESV2','Arlington']"     0     -1e+20          35          73
15   "Trans['RESV2','Medford']"       0         15          17          18
16   "Trans['RESV2','remainder']"     3         -3           0       1e+20
17   "Trans['RESV3','Cambridge']"     0     -1e+20          24          30
18   "Trans['RESV3','Boston']"        2         12          14       1e+20
19   "Trans['RESV3','Watertown']"     0     -1e+20          12          15
20   "Trans['RESV3','Everett']"       0     -1e+20          11          12
21   "Trans['RESV3','Malden']"        0         27          28          28
22   "Trans['RESV3','Arlington']"    38         38          76       1e+20
23   "Trans['RESV3','Medford']"       0         20          20       1e+20
24   "Trans['RESV3','remainder']"     0     -1e+20           0           3
;

:          _conname        _con.dual _con.down _con.current _con.up    :=
1    "Supply['RESV1']"         -12      1500         1700       2200
2    "Supply['RESV2']"          -3      2300         2500       3000
3    "Supply['RESV3']"           0      2400         2600      1e+20
4    "Demand['Cambridge']"      24         0          900       1100
5    "Demand['Boston']"         12       700         1200       1400
6    "Demand['Watertown']"      12         0          600        800
7    "Demand['Everett']"        11         0          400        600
8    "Demand['Malden']"         28      1000         1500       1700
9    "Demand['Arlington']"      38       500         1000       1200
10   "Demand['Medford']"        20       500         1000       1200
;
```

The sensitivity report is divided into two sections. The first section refers to the variables, with the columns giving the variable names, associated reduced cost, and range information on its cost coefficients. The second section refers to the constraints, with the columns giving the constraint names, associated dual solution, and range information for the constraint's right hand side.

Task 6

Answer the following questions using the second contractor's sensitivity report and the current optimal solution. Make sure you provide units in answering questions relating to quantities and are clear about the direction of the change.

If an answer cannot be determined based on the information from the sensitivity report, explain why not. Do not attempt to find the answer through other means, i.e. by re-solving the linear program.

Note 1: AMPL's definition of reduced cost is "the amount by which the objective function will be increased per unit increase in each non-basic variable", which is the opposite of the definition we have been using, where the reduced cost is "the amount by which the objective function will be decreased per unit increase in each non-basic variable". [Thus, you'll need to negate reduced cost values to get our convention!]

Note 2: This is a minimization problem. In interpreting the sensitivity information, the sign of all reduced cost and dual values is exactly the opposite of what we would expect for a maximization problem. Make sure the direction of change you write down within a solution makes intuitive sense!

1. What happens to the optimal total cost if the supply at reservoir 2 increases by 300 million gallons?
2. What happens to the optimal total cost if the supply at reservoir 1 decreases by 50 million gallons?
3. What happens to the optimal total cost if the supply at reservoir 3 increases by 1000 million gallons?
4. What happens to the optimal total cost if the demand at Malden decreases by 200 million gallons?
5. What happens to the optimal total cost if the demand at Watertown increases by 150 million gallons?
6. Assume that the cost of transport from reservoir 2 to Cambridge increased by 5 units ($500). Is the current basis still optimal? If so, what's the new objective value?
7. Assume that the cost of transport from reservoir 1 to Medford increased by 2 units ($200). Is the current basis still optimal? If so, what's the new objective value?

End Task 6

Having performed your public duty honorably, you decide to kick back and look into the first contractor's report as well.

Task 7

1. Read `transp1.mod` and `water1.dat`. Add comments to each line in `transp1.mod`. Make sure your comments explain how imbalances in supply and demand (supply > demand) is dealt with.
2. Write down the mathematical formulation from `transp1.mod`. Convert the formulation to its dual form. You do not have to implement the dual model in AMPL, nor write down its associated data file. You also do not have to explain the math formulation in this subtask.
3. Give a real-world interpretation of the dual formulatio of this problem: what could the varibles, constraints and objective represent? Your interpretation should be a consistent story that relates to all elements of the dual formulation.

End Task 7

5 Turning in your assignment

Final Task 8

You must turn in your assignment by 5:00 PM, Friday, October 14, 2016 [**No late days**].

For AMPL exercises, this includes any model and data files you have created. When writing down the solution from AMPL, always include both the objective value and the values assigned to variables (when the program is feasible and bounded, of course!)

Gather the AMPL model and data files, as well as a script file containing the AMPL commands you used to solve the problems. Upload your files to Canvas as a PDF of the solution and a zip file of AMPL/data files before 5:00 PM, Friday, October 14, 2016 [**No late days**]. Please scan your homework if you completed it on paper (use a scanner app not a camera if using your smartphone).

End Task 8

Congratulations on completing your fourth AM/ES 121 assignment!