# Problem Set 2
# Geometry, Algebra, Reality

AM121/ES121 — Fall 2017

Due 5:00 PM, Friday, September 22, 2017

## Announcements

- The assignment is due by 5:00 PM, Friday, September 22, 2017.
- Please type or scan your assignment and submit it to the Canvas folder for this assignment.
- Readings: (Recommended) Jensen and Bard, sections 3.1 to 3.3; Bradley, Hax, and Magnanti, Appendix A (available online; see link in syllabus).

## Goals

This assignment has three goals. First, you will explore the relationship between the algebra and geometry behind linear programs. Second, you will sharpen your modeling skills by formulating linear programs for real world examples. Finally, you will learn to use AMPL to express and solve linear programs on the computer.

## Contents

# 1 Optimality, unboundedness, and infeasibility

Consider the following linear program:

$$\begin{array}{rr} \text{minimize} & c_1x_1 + c_2x_2 \\ \text{subject to} & -3x_1 + 4x_2 \leq 5 \\ & x_1 + 2x_2 \leq 2 \end{array}$$

for nonzero constants $c_1$ and $c_2$.

**Task 1**

1. Find an example of values for $c_1$ and $c_2$ such that the objective value is unbounded.
2. Find an example of values for $c_1$ and $c_2$ such that the problem has multiple optimal solutions.
3. Find an example of values for $c_1$ and $c_2$ such that the problem has a single optimal solution.
4. Add a constraint to the problem so that the LP has no feasible solutions.

**End Task 1**

# 2 Extreme points and basic solutions

Consider the set of vectors $x$ satisfying the following constraints:

$$\begin{aligned} x_1 + 3x_2 + 2x_3 + 5x_4 &= 5 \\ 2x_1 + 6x_2 + 5x_3 + 3x_4 &= 9 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

For the following exercises, be sure to justify your answers.

**Task 2**

1. List the basic solutions corresponding to these equations. For each basic solution you write down, specify the columns of the coefficient matrix corresponding to the solution. Which ones are feasible?
2. Find an extreme point of the feasible region.
3. Find a basic solution that is not feasible.
4. Find a feasible solution that is not basic.
5. Find a feasible solution that is not an extreme point.

**End Task 2**

# 3 Learning to pivot

The Simplex algorithm iterates from tableau to tableau by performing pivot operations. Without getting into what it means to pivot on a tableau, in this exericse we get some initial practice with *pivoting on a* *matrix*. We can think about this as the $m$-by-$n$ matrix $\bar{A}$, while the tableau in Simplex also includes the right hand side $\bar{b}$ and the objective equation. Thus this is not the full pivot operation of Simplex.

Consider the following matrix:

$$\begin{pmatrix} 2 & 2 & 1 & 5 \\ 1 & 2 & 2 & 6 \\ -3 & 4 & 5 & 4 \end{pmatrix}$$

**Task 3**

1. Using 3 pivot operations, arrive at a matrix in canonical form (i.e., with three unit column vectors). You may wish to use mathematical software (e.g., MATLAB) to avoid hand calculations.
2. Find three additional, canonical-form matrices, by making three successive, additional pivots from the matrix found in task 3.1.

**End Task 3**

# 4 Non-linear objectives to linear objectives

Consider an optimization problem with variables $x = (x_1, \dots, x_n)$ of the following form:

$$\begin{array}{rc} \min\limits_{x} & \sum_{i=1}^{n} f_i(x_i) \\ \text{subject to} & x \in S \end{array}$$

here $f_i(x_i)$ is a non-negative function for which $f_i(x_i) = \beta_i^+ x_i$ when $x_i \geq 0$ and $f_i(x_i) = -\beta_i^- x_i$ when $x_i < 0$ for all $i$, and $\beta_i^+$ and $\beta_i^-$ are positive constants. $S$ denotes the feasible region.

**Task 4**

Assume the feasible region $S$ is defined by a set of linear constraints. Transform the optimization problem above into an equivalent linear programming formulation. Explain and justify your answer.

**End Task 4**

# 5 Making paper

A paper production company uses a combination of different kinds of raw wood fibers to produce paper. Each type of wood fiber has a different purity rating, is available in a different amount and at a different cost:

| Wood Fiber | Purity | Quantity available (kg) | Cost \$/kg |
|---|---|---|---|
| residue | 40 | 6000 | 0.05 |
| recycled fiber | 80 | 4000 | 0.40 |
| round wood | 100 | 1000 | 0.60 |

By using different combinations of the three raw materials, the company is able to make three types of paper, where the grade of paper is the weighted average of the purities of the wood fibers that it is composed of. The papers are sold at the following prices:

| Paper | Minimum Paper Grade | Selling price $/kg |
|---|---|---|
| Premium photo | 90 | 1.00 |
| Photocopy | 70 | 0.50 |
| Wrapping | 45 | 0.25 |

Table 1: Minimum amounts of solution required

| | | |
|---|---|---|
| 27 | 15 | 40 |
| 30 | 20 | 20 |
| 17 | 40 | 32 |

**Task 5**

1. Define the sets, parameters, and variables relevant to the problem.
2. Formulate a generalized *linear programming* model to find the production plan that maximizes the company's profit. Your formulation should be correct for any possible input values for the parameters and not just the ones given. You may wish to write a few sentences to explain parts of your model that aren't straightforward.
3. Write the corresponding model and data files in AMPL and solve the linear program (for the given parameter values). What should the company do?

**End Task 5**

# 6 Inorganic farming

Taste Matters, an inorganic producer of fruits and vegetables, has hired you to assist in configuring their plant watering system over their experimental test plantation.[1] The system consists of a series of sprinklers above and around the test plantation, as shown in the following diagrams they've provided:

| | | | | |
|---|---|---|---|---|
| s | s | s | s | s |
| s | s | s | s | s |
| s | s | s | s | s |
| s | s | s | s | s |
| s | s | s | s | s |

(a) Sprinkler

| | | | | |
|---|---|---|---|---|
| e | e | e | e | e |
| e | p | p | p | e |
| e | p | p | p | e |
| e | p | p | p | e |
| e | e | e | e | e |

(b) Plants

Figure 1: Plant and sprinkler system diagrams. $s$ indicates a sprinkler, $p$ indicates a plant, and $e$ indicates a piece of land without any plants.

The sprinklers and plants are evenly spaced, with all sprinklers are placed 7 feet off the ground, and with the inner sprinklers placed directly above the plants. However, Taste Matters acknowledges that placing sprnklers directly above plants has turned out to be a bad decision, as watering straight down onto the plant at full force may damage the plant. Thus, they have set each sprinkler on uniform spray over a 3 by 3 region centered over the plant below at the following concentration:

| | | |
|---|---|---|
| 0.1 | 0.1 | 0.1 |
| 0.1 | 0.1 | 0.1 |
| 0.1 | 0.1 | 0.1 |

For example, if the center most sprinkler sprays 25mL of the solution, 2.5mL of the solution will reach every single plant. As another example, if the sprinkler over the upper left most plant sprays 25mL of solution, 2.5mL of the solution will reach the four plants in the upper left corner.

Given these details, Taste Matters wishes you to come up with a `minimum cost schedule` for spraying the plants, given that each plant receives the minimum amount of solution it requires (in mL), as given by Table 1.

You tell Taste Matters that there are many ways to define a `minimum cost function`, and they tell you that of course there are, but you shouldn't worry about that right now since you will get to that in Task 7.

**Task 6**

Define the sets, parameters, and variables relevant to the problem.

**End Task 6**

**Task 7**

For the following parts, formulate the corresponding `linear program` models using the set, parameter, and variable definitions from the previous exercise. Your formulation should be correct for any possible input values for the parameters and not just the ones given. You may wish to write a few sentences to explain parts of models that aren't straightforward. For parts after the first, you can specify the necessary modifications to the model instead of rewriting the entire model.

1. First off, Taste Matters wants you to find a sprinkling policy to minimize the amount of solution used for spraying the plants.
2. After seeing your model, Taste Matters wants you to modify it to better distribute the sprinkling over the sprinklers. More specifically, because the entire sprinkling system needs to be active as long as one sprinkler is running, they wish you to minimize the maximum amount of solution from any particular sprinkler.
3. "We didn't mean for you to only focus on balancing," Taste Matters says. "You should use a linear combination to both minimize the total amount of solution and minimize the maximum amount from a particular sprinkler. You will reformulate the objective, and for now just weigh the two objectives equally while maintaining the scaling of the original objectives."
4. Taste Matters is happy with your work, so they give you more work. They wish you to model the cost of overwatering a plant, which not only wastes solution, but could seriously damage the plants. For each mL overwatered, the cost (waste and damage) is approximately \$$w$. They wish you to minimize the total cost for overwatering.
5. "We like to perform an experiment," Taste Matters says, "where we care not about cost, but the accuracy of providing exactly the amount of solution that each plant demands. Overwatering and underwatering is just as bad, for the purpose of this experiment. Find us a way to do this."

**End Task 7**

> Food for thought: Does your model account for cases where the plants are not surrounded by a border of sprinklers? (*E.g.*, the plants and sprinklers are on arbitrary locations in a grid.) It doesn't have to for this assignment, but Taste Matters may just need it for the next assignment.

[1] It's not water that's coming out, but they won't tell you exactly what it is they use!

# 7 Turning in your assignment

**Final Task 8**

You must turn in your assignment by 5:00 PM, Friday, September 22, 2017. Remember you may only use 1 late day per assignment. This means that the latest deadline for this homework is Saturday at 5pm. For AMPL exercises, this includes any model and data files you have created. When writing down the solution from AMPL, always include both the objective value and the values assigned to variables (when the program is feasible and bounded, of course!)

Gather the AMPL model and data files you have created for this assignment, as well as a script file containing the AMPL commands you used to solve the problems. Submit it to the Canvas folder for this assignment before 5:00 PM, Friday, September 22, 2017. Please scan and submit your homework if you completed it on paper. If you feel it may be hard to read, feel free to bring the original to class in addition to submitting online by the deadline.

**End Task 8**

Congratulations on completing your second AM121/ES121 assignment!