# Problem Set 1
# Getting Started with Linear Programming

AM121/ES121 — Fall 2017

Due 5:00 PM, Friday, September 15, 2017

## Announcements

- A reminder that sections will begin on the week of September 11. Please make sure you have filled out the online sectioning form (we have posted the link to Piazza). Office hours have already started and you can find the schedule on the AM121 website.
- Submit your problem set by uploading your solution write-up to the canvas Dropbox for this assignment. Please make sure to include a screenshot or the text of your data and code files in your writeup. Please upload a single file. If you have more than one file, zip the files.
- We would prefer typed assignments (and LaTeX is a very nice way to do this with math equations once you get the hand of it), but we won't insist on this. We will ask you to type them if the course staff cannot read your assignments. We'll insist on LaTex for the two extreme optimization papers (one submitted per group). Check out the course website for some LaTex resources. Feel free to post on Piazza withLaTeX questions or code!
- Readings: The AMPL book, Introduction and Ch 1. (Optional reference reading: Jensen and Bard - Ch 2, skip Ch 2.4).

## Goals

This assignment has four goals. First, you will practice formulating optimization problems. Second, you will explore the definition of a linear program and its possible solutions. Then, you will formulate models for a real-world problem using linear programming. Finally, you will use the AMPL software you setup on your computer.

## Contents

# 1 Cutting Cake

The purpose of the following exercise is to get you started with taking a real-world problem and formulating it into an optimization problem. Note that this exercise (and all modeling exercises) is more about formulating a general mathematical model than about the solution to the optimization problem given specific inputs. *Your mathematical models may not always be linear in this exercise.*

As with all modeling exercises, expressing your models clearly and precisely is essential. It is important that you read through the (short) AM121/ES121 mathematical programming style guide on the resources section of the course website before attempting this task and the rest of this course! Continue to refer to this guide for future problem sets. If you have a question, feel free to post it on Piazza.

**Task 1**

The AM121/ES121 teaching staff is planning to reward its students on the day of presentations for Extreme Optimization by serving cake. They are faced with two delicious desserteries: Flour and Toscanini's. Each store charges a certain amount per unit of cake and can create cakes of any volume with uniform heights of one unit. Thus, the staff needs only consider their cake options using two-dimensional models. Cakes from Flour are circular, while cakes from Toscanini's are rectangular. Because the teaching staff feels very strongly about equality, they demand that regardless of where the cake is from, all who attend must receive precisely the same amount of cake.

Both stores can deliver cakes pre-cut. Flour can make cuts along any diameter of the cake, while Toscanini's can make length-wise and width-wise cuts. Both of the stores can only make complete cuts (i.e., any cut must completely separate two sides of the cake) and charge a small fee for each cut desired. Toscanini's charges the same amount for length-wise and width-wise cuts. Let $F_f$ and $F_t$ denote these fees at Flour and Toscanini's, respectively.

1. The teaching staff first wants to know how it can maximize the number of people served. Let $B_c$ be the total amount of money that the teaching staff is willing to spend on cutting the cake. If they decide to order a circular cake from Flour, what is the maximum number of equally-sized slices that they can have? Please state your answer in terms of a mathematical program (you may wish to refer to the style guide), as well as with a closed-form solution.

2. Alternatively, the staff considers how to maximize the number of equally-sized slices given the same cutting budget if they order a rectangular cake from Toscanini's. Formulate an optimization problem and solve it using results from single-variable calculus. You may cite the results of certain derivatives, but please show your work. How can you make the best use of your budget? How many cuts should be made length-wise and how many cuts should be made width-wise? Double-check that your solution is fully specified for all possible parameter values.

3. Now suppose that the teaching staff knows exactly how many people will be attending the presentations, and they have been given a generous budget to purchase one cake under the condition that no slices are wasted. If all students must be given an equal amount of cake and all cutting must be professionally done, what is the set of numbers of students for which the staff may order from Flour and the set of numbers for which they may order from Toscanini's? Are there conditions under which the students will get no cake? Explain.

4. Now suppose that the teaching staff is given a total budget $B$ for buying and cutting a cake from the desserterie of their choice, without the constraint that no cake may go wasted (i.e., leftovers are allowed). If $n$ people will be attending the presentations, how should the teaching staff decide between the two desserteries if they wish to maximize the total amount of cake per student? You should write down two fully-specified mathematical programs and discuss how you would use the solutions to these programs to reach your answer. Do not solve the optimization problems.

5. Upon solving these two optimization problems after inputting the parameters they are faced with, the course staff notices that round cakes from Flour would be the optimal choice. While on their way to Flour to place their order, the course staff passes by Toscanini's. Kevin notices a sign on the outside of Toscanini's which reads, "Due to technical difficulties, we must limit the number of width-wise cuts on any given cake to 7." Given this new information, what is the best thing for the course staff to do? Explain your answer in terms of the mathematical programs you formulated above.

6. What factors, not captured in the current models, may influence the course staff's final decision regarding which desserterie they should ultimately select?

**End Task 1**

# 2 LP or Not?

**Task 2**

For each of the following examples, determine whether the model is a linear program. If not, why not?

**End Task 2**

1.

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 5x_2 + 4x_1x_2 \\ \text{subject to} \quad & 3x_1 + 6x_2 \le 10 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \end{aligned}$$

2.

$$\begin{aligned} \text{maximize} \quad & 3x_1 + 7x_2 \\ \text{subject to} \quad & 5x_1^2 + 6x_2 \le 11 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \end{aligned}$$

3.

$$
\begin{aligned}
\text{minimize} \quad & 2x_1 + 3x_2 - 6x_7 - x_4 \\
\text{subject to} \quad & x_1 + x_4 \geq 10 \\
& x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_7 = 0 \\
& x_4 \geq 0
\end{aligned}
$$

4.

$$
\begin{aligned}
\text{minimize} \quad & |x_1| + |x_2| \\
\text{subject to} \quad & x_1 + x_2 \geq 10
\end{aligned}
$$

## 3 These are LPs. But...

Not every linear program has an optimal solution. Some may be unbounded (objective value going off to infinity), and others may be infeasible (no values for the variables can satisfy all the constraints). Consider the following linear programs:

1.

$$
\begin{aligned}
\text{maximize} \quad & -x_2 \\
\text{subject to} \quad & x_1 + x_2 \geq 4 \\
& x_1 - x_2 \leq -2 \\
& x_2 \geq 2
\end{aligned}
$$

2.

$$
\begin{aligned}
\text{maximize} \quad & -x_2 \\
\text{subject to} \quad & x_1 + x_2 \geq 4 \\
& x_1 - x_2 \leq -2 \\
& x_2 \leq 2
\end{aligned}
$$

3.

$$
\begin{aligned}
\text{maximize} \quad & x_2 \\
\text{subject to} \quad & x_1 + x_2 \geq 4 \\
& x_1 - x_2 \leq -2 \\
& x_2 \geq 2
\end{aligned}
$$

**Task 3**

For each of these linear programs, determine graphically whether the program has an optimal solution, is infeasible, or is unbounded. Try using computer software to graph the regions, e.g., see our Mathematica graphing guide, or try Wolfram Alpha.

**End Task 3**

**Task 4**

Explain algebraically why your answers to the above task are correct.

**End Task 4**

## 4 Find the optimal solution

Consider the following linear program:

$$\begin{aligned} \text{maximize} \quad & 7x_1 + 6x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq 5 \\ & x_1 + 3x_2 \leq 16 \\ & 2x_1 + x_2 \leq 12 \\ & x_1, x_2 \geq 0 \end{aligned}$$

**Task 5**

1. Find the optimal feasible solution graphically.
2. Prove algebraically that your answer to the above task is optimal.

**End Task 5**

## 5 Made by me

"Made By Me" is a company that produces three types of ceramics: bowls, cups, and vases. Making a product involves 3 steps: molding, firing, and glazing. The speed of each step of production for each type of ceramics, in number of ceramic pieces per hour, are as follows:

| | vases | bowls | cups |
|---|---|---|---|
| molding | 1 | 2 | 5 |
| firing | 3 | 2 | 4 |
| glazing | 3 | 4 | 2 |

Due to production limits, the firm can allocate at most 10 hours to molding, 8 hours to firing, and 7 hours to glazing. The firm makes \$20 per vase, \$30 per bowl, and \$40 per cup, and wishes to maximize profit.

**Task 6**

As with all modeling exercises, expressing your models clearly and precisely is essential. Please refer to the AM121/ES121 mathematical programming style guide on the resources section of the course website. If you have a question, feel free to post it on the Piazza board.

1. Formulate this problem as a linear program model. Be clear about what the variables are, what the objective function is, and what the constraints are. Define parameters and sets to generalize your model: your formulation should be correct for any possible input values for the parameters and not just the ones given. Also specify the values of the parameters for the given input.
2. You realize that the firm has neglected to consider costs in maximizing profit. The firm apologizes for their mistake, and lets you know that each vase costs 6 dollars to produce, each bowl costs 9 dollars to produce, and each cup costs 11 dollars to produce. Update your model to take costs into account. Make sure your model is still a linear program.

3. The firm thanks you for your work, and now wishes you to take demand into account as to not produce more units than can be sold. Demand is 35 units for bowls, 30 units for cups, and 45 units for vases. Update your model to take demand into account. Make sure your model is still a linear program. By inspection, do you think this constraint will affect the optimal solution for the data in this particular example? Justify your answer.
4. While the demand constraints allow the firm not to over-produce, the firm wishes also to keep production balanced as not to produce too much of any particular item. As such, the firm wishes you to update the model as to allow for no more than 45% of production to be of any particular item. Update your linear programming model to take balancing into account. Hint: write down the inequality that corresponds to the condition you need. If it is linear, you are good. If it isn't, can you make it linear?

**End Task 6**

# 6 Cooling down with AMPL

For reference, you may want to read the Introduction and Chapter 1 of the AMPL book, available at *http://ampl.com/BOOK/contents.html*. The model and data files that Chapter 1 refers to are linked on the bottom of the assignments page of the course website. Follow the chapter, trying out commands on the computer as you go. The information covered here will not only help you with the AMPL software, but also give you a gentle introduction into linear programming. Don't worry too much about the discussion on "marginal values" (starting on p. 17). And you can skip section 1.7.

Quark has recently got in on the highly lucrative video game industry. With Rom's help, Quark can produce 3 types of video game consoles: "Nintendo Wii", "Playstation 3", and "Xbox 360". He can produce the Wii at 25,000 consoles per hour, for a profit of $20 per console; he can produce the PS3 at 20,000 consoles per hour, for a profit of $15 per console; and he can produce the 360 at 15,000 consoles per hour, for a profit of $20 per console. Quark operates 18 hours a day, and can ship at most 130,000 consoles of each type of console each day. Quark has no loyalties to any particular console and only wishes to maximize profit. Quark reminds you that shipping fractional consoles units is allowed (hey, nothing in the contract regulations say anything about enforcing integer constraints!)

**Task 7**

1. Read through the model file `prod.mod` (the download link is on the assignments page) This is the same `prod.mod` you read about in the first chapter of the AMPL book. Copy this file to your working directory - the AM121/ES121 folder you created as part of the AMPL set up.
2. Encode the specifics of Quark's problem as parameter values for the production model. Create an AMPL data file called `console.dat` with these values that is compatible with the model in `prod.mod`. Use AMPL to solve the problem. How much of each console should Quark produce? How much profit will he make?

**End Task 7**

# Turning in your assignment

All of the problem sets in AM121/ES121 will have a "Final task" that summarizes what you should submit to the course staff. Follow the directions below to submit your first assignment.

**Final Task 8**

The due date of this assignment is 5:00 PM, Friday, September 15, 2017. Remember you may only use 1 late day per assignment. This means that the latest deadline for this homework is Saturday at 5pm. Submit your pset by uploading your solution write-up to the canvas Dropbox for this assignment. Please make sure to include a screenshot or the text of your data file `console.dat` from the last problem in your writeup. (Please upload a single file. If you have more than one file, zip the files.)

Please scan and submit your homework as a single PDF file if you completed it on paper. If you feel it may be hard to read, feel free to bring the original to class in addition to submitting online by the deadline.

Congratulations on completing your first AM121/ES121 assignment!

**End Task 8**